

CATEGORY \mathcal{O} FOR RATIONAL CHEREDNIK ALGEBRAS $H_{t,c}(GL_2(\mathbb{F}_p), \mathfrak{h})$ IN CHARACTERISTIC p

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ABSTRACT. In this paper we describe the characters of irreducible objects in category \mathcal{O} for the rational Cherednik algebra associated to $GL_2(\mathbb{F}_p)$ over an algebraically closed field of positive characteristic p , for any value of the parameter t and generic value of the parameter c .

1. INTRODUCTION

This paper is a sequel to [1], and continues the study of the rational Cherednik algebras over algebraically closed fields of positive characteristic.

Let p be an odd prime, \mathbb{k} an algebraically closed field of characteristic p , $\mathbb{F}_p \subset \mathbb{k}$ the subfield of p elements, $G = GL_2(\mathbb{F}_p)$ the general linear group over \mathbb{F}_p , $\mathfrak{h} = \mathbb{k}^2$ the tautological representation of G , $\mathfrak{h}_{\mathbb{F}}$ its \mathbb{F}_p -form, and \mathfrak{h}^* the dual representation. For $t \in \mathbb{k}$ a constant, and c a collection of conjugation invariant parameters in \mathbb{k} labeled by reflections in G , the *rational Cherednik algebra* $H_{t,c}(G, \mathfrak{h})$ is a non-commutative, associative, infinite-dimensional algebra deforming the semidirect product of the group algebra $\mathbb{k}[G]$ and the symmetric algebra $S(\mathfrak{h}^* \oplus \mathfrak{h})$. Algebras of this type, for various reflection groups G , have been extensively studied since the early 1990s, mostly over fields of characteristic zero.

We first repeat some general results about rational Cherednik algebras (which can be found in [4]), and some results specific for finite characteristic (which can be found in [1]). In particular, we restate the definition of category \mathcal{O} of $H_{t,c}(G, \mathfrak{h})$ -representations, standard or Verma modules $M_{t,c}(\tau)$ parametrized by the irreducible G -representations τ , the contravariant form B on $M_{t,c}(\tau)$, and the irreducible modules $L_{t,c}(\tau) \cong M_{t,c}(\tau) / \text{Ker } B$ in category \mathcal{O} . In characteristic p , the irreducible modules $L_{t,c}(\tau)$ are always finite-dimensional, and for generic value of the parameter c they have a specific form depending on the structure of a *reduced module*. We define characters and reduced characters, and discuss baby Verma modules $N_{t,c}(\tau)$, which are the quotients of $M_{t,c}(\tau)$ by the action of a large central subspace of $H_{t,c}(G, \mathfrak{h})$. Up to grading shifts, all irreducible objects in \mathcal{O} are isomorphic to some $L_{t,c}(\tau)$.

Next, we state some results about the structure of the category of finite dimensional representations of $GL_2(\mathbb{F}_p)$. These are finite reflection groups which have no counterpart in characteristic zero, and have the property that the characteristic divides the order of the group. We summarize some results about their representations which we use in later computations.

The main part of the paper is calculating the characters of irreducible $H_{t,c}(GL_2(\mathbb{F}_p), \mathfrak{h})$ -modules $L_{t,c}(\tau)$, for generic c and all τ . The main theorem is Theorem 4.1.

The roadmap of this paper is as follows. Section 2 contains definitions and basic properties of rational Cherednik algebras, their representations (Verma, baby Verma and irreducible modules), and category \mathcal{O} . We repeat most results without proofs and stress out the differences between characteristic zero and characteristic p case. In section 3 we discuss the

properties of the group $GL_2(\mathbb{F}_p)$ and its representations which directly influence the structure of rational Cherednik algebras and their representations. Section 4 contains some preliminary observations and calculations about category \mathcal{O} for $H_{t,c}(GL_2(\mathbb{F}_p), \mathfrak{h})$, including the observation that characters for $L_{t,c}(\tau)$ can be calculated for all $p^2 - p$ irreducible representations τ from the character formulas for p specific representations τ . Finally, sections 5-7 contain character calculations for irreducible representations associated to these p representations of $GL_2(\mathbb{F}_p)$.

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2. RATIONAL CHEREDNIK ALGEBRAS AND THEIR REPRESENTATIONS

2.1. Notation. Let \mathbb{k} be an algebraically closed field, \mathfrak{h} an n -dimensional \mathbb{k} -vector space, and $G \subseteq GL(\mathfrak{h})$ a finite group generated by the set S of reflections in G . An element $s \in G$ is a reflection if rank of $1 - s$ on \mathfrak{h} is 1. Let \mathfrak{h}^* be the dual representation, and (\cdot, \cdot) the canonical pairing $\mathfrak{h} \otimes \mathfrak{h}^* \rightarrow \mathbb{k}$ or $\mathfrak{h}^* \otimes \mathfrak{h} \rightarrow \mathbb{k}$. Let $\alpha_s \in \mathfrak{h}^*$ be the basis of the image of $1 - s$ on \mathfrak{h}^* .

For most of this paper, we will let G be the general linear group over a finite field with p elements, and \mathfrak{h} the tautological representation. We first list the definitions and properties that do not depend on the characteristic nor on the choice of G , and then restrict our attention to characteristic p and $G = GL_2(\mathbb{F}_p)$.

Let us introduce some notation: for a \mathbb{k} -vector space V , let TV and SV denote the tensor and symmetric algebra of V over \mathbb{k} , and $S^i V$ the homogeneous subspace of SV of degree i . For a graded vector space M , let M_i denote the i -th graded piece, and $M[j]$ the same vector space with the grading shifted by j , meaning $M[j]_i = M_{i+j}$. For $M = \bigoplus_i M_i$ a graded vector space, define its Hilbert series as $h(z) = \sum_i \dim M_i z^i$.

The following definitions and results are standard and can be found in [4].

2.2. Rational Cherednik algebras. Let $t \in \mathbb{k}$, and $s \mapsto c_s$ be a \mathbb{k} -valued function on the set S of reflections in G , satisfying $c_s = c_{gsg^{-1}}$ for any $g \in G$.

Definition 2.1. The *rational Cherednik algebra* $H_{t,c}(G, \mathfrak{h})$ is the quotient of the semidirect product $\mathbb{k}[G] \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$ by the ideal generated by relations:

$$[x, x'] = 0, [y, y'] = 0, [y, x] = (y, x)t - \sum_{s \in S} c_s ((1 - s) \cdot x, y)s,$$

for all $x, x' \in \mathfrak{h}^*, y, y' \in \mathfrak{h}$.

For $g \in G$ and $y \in \mathfrak{h}$, we use notation gy for multiplication in the algebra, and $g \cdot y$ for the action from the representation; they are related by $gyg^{-1} = g \cdot y$.

The standard PBW theorem holds for $H_{t,c}(G, \mathfrak{h})$, independent on the characteristic (see [6], Theorem 2.1.).

Theorem 2.2 (PBW Theorem for Cherednik algebras). *The set*

$$\{gx_1^{a_1} \cdots x_n^{a_n} y_1^{b_1} \cdots y_n^{b_n} \mid g \in G, a_i, b_i \geq 0\}$$

is a basis for $H_{t,c}(G, \mathfrak{h})$.

For any $a \in \mathbb{k}^\times$ there exists an isomorphism of algebras $H_{t,c}(G, \mathfrak{h}) \cong H_{at,ac}(G, \mathfrak{h})$. Because of that, we can rescale the parameters (t, c) and assume without loss of generality that $t = 0$ or $t = 1$. These two cases behave differently, and we study them both.

2.3. Verma Modules $M_{t,c}(\tau)$ and Dunkl operators.

Definition 2.3. Let τ be an irreducible finite-dimensional representation of G . Define a $\mathbb{k}[G] \ltimes S\mathfrak{h}$ -module structure on it by requiring \mathfrak{h} -action on τ to be zero. The *Verma module* is the induced $H_{t,c}(G, \mathfrak{h})$ -module

$$M_{t,c}(G) = H_{t,c}(G, \mathfrak{h}) \otimes_{\mathbb{k}[G] \ltimes S\mathfrak{h}} \tau.$$

$M_{t,c}(\tau)$ satisfies the following universal mapping property:

Lemma 2.4. *Let M be an $H_{t,c}(G, \mathfrak{h})$ -module. Let $\tau \subset M$ be a G -submodule on which $\mathfrak{h} \subseteq H_{t,c}(G, \mathfrak{h})$ acts as zero. Then there is a unique homomorphism $\phi : M_{t,c}(\tau) \rightarrow M$ of $H_{t,c}(G, \mathfrak{h})$ -modules such that $\phi|_\tau$ is the identity.*

Define a grading on $H_{t,c}(G, \mathfrak{h})$ on generators by letting $\deg x = 1$, $\deg y = -1$ and $\deg g = 0$ for $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$, $g \in G$. This makes $H_{t,c}(G, \mathfrak{h})$ a graded algebra, and by additionally declaring $\tau \subseteq M_{t,c}(\tau)$ to have degree 0, it makes $M_{t,c}(\tau)$ a graded representation. More precisely, as vector spaces $M_{t,c}(\tau) \cong S\mathfrak{h}^* \otimes \tau$, and $M_{t,c}(\tau)_i \cong S^i \mathfrak{h}^* \otimes \tau$. In particular, each graded piece of $M_{t,c}(\tau)$ is a finite dimensional G -representation.

The action of the generators of $H_{t,c}(G, \mathfrak{h})$ on $M_{t,c}(\tau) \cong S\mathfrak{h}^* \otimes \tau$ can be explicitly written: for $f \otimes v \in S\mathfrak{h}^* \otimes \tau$, $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$ and $g \in G$, we have

$$x.(f \otimes v) = (xf) \otimes v,$$

$$g.(f \otimes v) = g.f \otimes g.v$$

$$y.(f \otimes v) = t\partial_y(f) \otimes v - \sum_{s \in S} c_s \frac{(y, \alpha_s)}{\alpha_s} (1 - s).f \otimes s.v.$$

The operators

$$D_y = t\partial_y \otimes 1 - \sum_{s \in S} c_s \frac{(y, \alpha_s)}{\alpha_s} (1 - s) \otimes s$$

are called *Dunkl operators*.

We say a homogeneous element $v \in M_{t,c}(\tau)$ is *singular* if $D_y v = 0$ for all $y \in \mathfrak{h}$. Any such element of positive degree generates a proper $H_{t,c}(G, \mathfrak{h})$ submodule. By Lemma 2.4, this submodule is isomorphic to a quotient of $M_{t,c}(G.v)$.

2.4. Contravariant Form B . For any graded $H_{t,c}(G, \mathfrak{h})$ -module $M = \oplus M_i$ with finite dimensional graded components M_i , define its restricted dual as $M^\dagger = \oplus M_i^*$. It is a left module for the opposite algebra $H_{t,c}(G, \mathfrak{h})^{opp}$. Define $\bar{c} : S \rightarrow \mathbb{k}$ as $\bar{c}_s = c_{s^{-1}}$. There is a natural isomorphism $H_{t,c}(G, \mathfrak{h})^{opp} \rightarrow H_{t,\bar{c}}(G, \mathfrak{h}^*)$ that is the identity on \mathfrak{h} and \mathfrak{h}^* , and sends $g \mapsto g^{-1}$ for $g \in G$, making M^\dagger a $H_{t,\bar{c}}(G, \mathfrak{h}^*)$ -module.

For τ an irreducible finite-dimensional representation of G , by Lemma 2.4, there is a unique homomorphism $\phi : M_{t,c}(\tau) \rightarrow M_{t,\bar{c}}(\tau^*)^\dagger$ which is the identity in the lowest graded piece τ . By adjointness, it is equivalent to the *contravariant form* pairing

$$B : M_{t,c}(\tau) \times M_{t,\bar{c}}(\tau^*) \rightarrow \mathbb{k}.$$

Proposition 2.5. *The contravariant form B satisfies the following properties.*

- a) *It is G -invariant: for $f \in M_{t,c}(\tau)$, $h \in M_{t,\bar{c}}(\tau^*)$, $B(g \cdot f, g \cdot h) = B(f, h)$.*
- b) *For $x \in \mathfrak{h}^*$, $f \in M_{t,c}(\tau)$, and $h \in M_{t,\bar{c}}(\tau^*)$, $B(xf, h) = B(f, D_x(h))$.*
- c) *For $y \in \mathfrak{h}$, $f \in M_{t,c}(\tau)$, and $h \in M_{t,\bar{c}}(\tau^*)$, $B(f, yh) = B(D_y(f), h)$.*
- d) *The form is zero on elements in different degrees: if $f \in M_{t,c}(\tau)_i$ and $h \in M_{t,\bar{c}}(\tau^*)_j$, $i \neq j$, then $B(f, h) = 0$.*
- e) *The form is the canonical pairing of τ and τ^* in the zeroth degree: for $v \in \tau \cong M_{t,c}(\tau)_0$, $f \in \tau^* \cong M_{t,\bar{c}}(\tau^*)_0$, $B(v, f) = (v, f)$.*

As B respects the grading, we can think of it as a collection of bilinear forms on finite-dimensional graded pieces. Let B_i be the restriction of B to $M_{t,c}(\tau)_i \otimes M_{t,\bar{c}}(\tau^*)_i$. By definition, $\text{Ker } B = \text{Ker } \phi$ is a submodule of $M_{t,c}(\tau)$. Singular vectors of positive degree in $M_{t,c}(\tau)$ are in $\text{Ker } B$, and so are the submodules generated by them. The forms B_i can be computed inductively, using the above proposition. In the initial stages of this project, we calculated them using MAGMA algebra software [2].

2.5. Irreducible modules $L_{t,c}(\tau)$. The importance of the form B lies in the following theorem:

Theorem 2.6. *Verma module $M_{t,c}(\tau)$ has a maximal proper graded submodule $J_{t,c}(\tau)$, equal to $\text{Ker } B$.*

In characteristic zero, the assumption that the submodule is graded is unnecessary, while in characteristic p it is crucial. The proof of the theorem can be found in [4] for characteristic zero, and in [1] in characteristic p .

Let

$$L_{t,c}(\tau) = M_{t,c}(\tau) / J_{t,c}(\tau)$$

be the minimal quotient of the Verma module. It satisfies:

Proposition 2.7. *For any τ , the module $L_{t,c}(\tau)$ is an irreducible graded $H_{t,c}(G, \mathfrak{h})$ -module. If characteristic of \mathbb{k} is positive, then $L_{t,c}(\tau)$ is finite dimensional.*

2.6. Category \mathcal{O} . As pointed out in the previous subsection, there are differences in description of $J_{t,c}(\tau)$ and $L_{t,c}(\tau)$ between characteristic zero and characteristic p situation. From now on, we assume \mathbb{k} to be algebraically closed field of characteristic p . The following definitions and results can be found in [1].

Definition 2.8. The category $\mathcal{O} = \mathcal{O}_{t,c}(G, \mathfrak{h})$ is the category of \mathbb{Z} -graded $H_{t,c}(G, \mathfrak{h})$ -modules which are finite-dimensional over \mathbb{k} .

Theorem 2.9. *The irreducible objects in \mathcal{O} are the modules $L_{t,c}(\tau)[i]$, for all irreducible G -representations τ and all possible grading shifts $i \in \mathbb{Z}$.*

2.7. Baby Verma modules $N_{t,c}(\tau)$. In characteristic zero, the modules $M_{t,c}(\tau)$ are irreducible for generic value of c , and the definition of category \mathcal{O} in that situation is such that it contains $M_{t,c}(\tau)$. In characteristic p , the algebra $H_{t,c}(G, \mathfrak{h})$ always has a large center, and $M_{t,c}(\tau)$ always has a large graded submodule. Also, as all $L_{t,c}(\tau)$ are finite dimensional and category \mathcal{O} is defined to contain only finite dimensional modules, $M_{t,c}(\tau)$ are not in \mathcal{O} . For these reasons, we sometimes prefer using baby Verma modules, defined below, instead of Verma modules.

Let $((S\mathfrak{h}^*)^G)_+$ denote the subspace of G -invariants in $S\mathfrak{h}^*$ of positive degree, and $((S\mathfrak{h}^*)^G)_+^p$ the space of p -th powers of elements of $((S\mathfrak{h}^*)^G)_+$. For $t = 0$, $(S\mathfrak{h}^*)^G$ is a central subalgebra, and so $((S\mathfrak{h}^*)^G)_+^p M_{0,c}(\tau)$ is a proper submodule of $M_{0,c}(\tau)$; for $t \neq 0$, $((S\mathfrak{h}^*)^G)^p$ is central and $((S\mathfrak{h}^*)^G)_+^p M_{t,c}(\tau)$ is a proper submodule of $M_{t,c}(\tau)$.

Definition 2.10. For $t \neq 0$, the *baby Verma module* $N_{t,c}(\tau)$ for the algebra $H_{t,c}(G, \mathfrak{h})$ is the quotient

$$N_{t,c}(\tau) = M_{t,c}(\tau) / ((S\mathfrak{h}^*)^G)_+^p M_{t,c}(\tau).$$

For $t = 0$, the *baby Verma module* $N_{0,c}(\tau)$ for the algebra $H_{0,c}(G, \mathfrak{h})$ is the quotient

$$N_{0,c}(\tau) = M_{0,c}(\tau) / ((S\mathfrak{h}^*)^G)_+ M_{0,c}(\tau).$$

Proposition 2.11. *$N_{t,c}(\tau)$ is graded and finite dimensional, and thus in category \mathcal{O} . The form B descends to it. The kernel of this form on $N_{t,c}(\tau)$ is the maximal proper submodule of $N_{t,c}(\tau)$. The quotient of $N_{t,c}(\tau)$ by this kernel is isomorphic to $L_{t,c}(\tau)$.*

2.8. Characters.

Definition 2.12. Let $K_0(G)$ be the Grothendieck ring of the category of finite dimensional representations of G over \mathbb{k} . For $M = \oplus_i M_i$ any graded $H_{t,c}(G, \mathfrak{h})$ module with finite dimensional graded pieces, define its character to be the power series in formal variables z, z^{-1} with coefficients in $K_0(G)$

$$\chi_M(z) = \sum_i [M_i] z^i,$$

and its Hilbert series to be

$$\text{Hilb}_M(z) = \sum_i \dim(M_i) z^i.$$

If M is in category \mathcal{O} , it is finite dimensional and its character is in $K_0(G)[z, z^{-1}]$.

Proposition 2.13. *The character of the Verma module $M_{t,c}(\tau)$ is*

$$\chi_{M_{t,c}(\tau)}(z) = \sum_{i \geq 0} [S^i \mathfrak{h}^* \otimes \tau] z^i, \quad \text{Hilb}_{M_{t,c}(\tau)}(z) = \frac{\dim(\tau)}{(1-z)^n}.$$

For B_i the restriction of the contravariant form B to $M_{t,c}(\tau)_i \cong S^i \mathfrak{h}^* \otimes \tau$, the character of the irreducible module $L_{t,c}(\tau)$ is

$$\chi_{L_{t,c}(\tau)}(z) = \sum_{i \geq 0} ([S^i \mathfrak{h}^* \otimes \tau] - [\text{Ker } B_i]) z^i, \quad \text{Hilb}_{M_{t,c}(\tau)}(z) = \sum_{i \geq 0} \text{rank } B_i z^i.$$

Characters of baby Verma modules at $t = 0$ and $t = 1$ are related by

$$\chi_{N_{1,c}(\tau)}(z) = \chi_{N_{0,c}(\tau)}(z^p) \left(\frac{1 - z^p}{1 - z} \right)^n.$$

If G has the property that the algebra of invariants $(S\mathfrak{h}^*)^G$ is a polynomial algebra with homogeneous generators of degrees d_1, \dots, d_n , then the character and the Hilbert series of a baby Verma module $N_{t,c}(\tau)$, for $t = 0, 1$, is:

$$\begin{aligned} \chi_{N_{0,c}(\tau)}(z) &= \chi_{M_{0,c}(\tau)}(z)(1 - z^{d_1})(1 - z^{d_2}) \dots (1 - z^{d_n}). \\ \chi_{N_{1,c}(\tau)}(z) &= \chi_{M_{1,c}(\tau)}(z)(1 - z^{pd_1})(1 - z^{pd_2}) \dots (1 - z^{pd_n}). \\ \text{Hilb}_{N_{0,c}(\tau)}(z) &= \dim(\tau) \frac{(1 - z^{d_1})(1 - z^{d_2}) \dots (1 - z^{d_n})}{(1 - z)^n}. \\ \text{Hilb}_{N_{1,c}(\tau)}(z) &= \dim(\tau) \frac{(1 - z^{pd_1})(1 - z^{pd_2}) \dots (1 - z^{pd_n})}{(1 - z)^n}. \end{aligned}$$

The matrix of the form B_i on $M_{t,c}(\tau)_i$ depends polynomially on c . Combining this and the above proposition, we see that the character of $L_{t,c}(\tau)$ is the same for all generic c (meaning, c outside of finitely many hypersurfaces in the space of functions from the finite set of conjugacy classes in G to the ground field \mathbb{k}), while for special c (in this finite set of hypersurfaces), $\text{Ker } B_i$ might be larger. In this paper, we will be interested only in characters at generic value of c .

Proposition 2.14. (1) For $t \neq 0$ and generic c , the character of $L_{t,c}(\tau)$ is of the form

$$\chi_{L_{t,c}(\tau)}(z) = \chi_{S^{(p)}\mathfrak{h}^*}(z)H(z^p),$$

where $S^{(p)}\mathfrak{h}^*$ is the quotient of $S\mathfrak{h}^*$ by the ideal generated by x_1^p, \dots, x_n^p , and $H \in K_0[z]$ is a character of some graded G -representation. In particular, the Hilbert series of $L_{t,c}(\tau)$ is of the form

$$\text{Hilb}_{L_{t,c}(\tau)}(z) = \left(\frac{1 - z^p}{1 - z} \right)^n \cdot h(z^p),$$

for h a polynomial with nonnegative integer coefficients.

(2) The polynomial h satisfies $1 \leq h(1) \leq |G|$, and $\dim L_{t,c}(\tau) = h(1)p^n$.

Definition 2.15. We call H the reduced character, and h the reduced Hilbert series of $L_{t,c}(\tau)$.

The idea of the proof of (1) is to construct a subrepresentation $J'_{t,0}(\tau)$ of $M_{t,0}(\tau)$ which behaves as if $c = 0$ were a generic point, even when it is not. More specifically, we pick a line through the origin in the space of all possible parameters c (a \mathbb{k} vector space of dimension equal to the number of conjugacy classes), such that all but finitely many points on this line are generic values of the parameter for the rational Cherednik algebra. As a consequence, the modules $J_{t,c}(\tau)$ for all generic c on this line have the same Hilbert series, and their graded pieces $J_{t,c}(\tau)_i$, which are representations of the group, all have the same composition series. Associating to each generic c the representation $J_{t,c}(\tau)$ can be seen as a map from a punctured \mathbb{k} -line to the appropriate Grassmanian. We define $J'_{t,0}(\tau)$ to be the extension of this map to $c = 0$, and think of it as a limit of $J_{t,c}(\tau)$ as c goes to zero.

This space $J'_{t,0}(\tau)$ has the following properties:

- It is an $H_{t,0}(G, \mathfrak{h})$ subrepresentation;

- It has the same Hilbert series as $J_{t,c}(\tau)$ for generic c ;
- It is contained in $J_{t,0}(\tau)$ (properly contained if $c = 0$ is not a generic point);
- It is G -invariant, and every graded piece $J'_{t,0}(\tau)_i$ has the same composition series as $J_{t,c}(\tau)_i$ for generic c .

As a consequence, the character of $L_{t,c}(\tau)$ at generic c is the same as the character of $M_{t,c}(\tau)/J'_{t,0}(\tau)$. On the other hand, $J'_{t,0}(\tau)$ is stable under Dunkl operators, which, at $c = 0$ and $t \neq 0$, are multiples of ∂_y . It is easy to show that because of that, $J'_{t,0}(\tau)$ is generated by p -th powers, meaning by elements of the form $f^p \otimes v$ for some $f \in S\mathfrak{h}^*$, $v \in \tau$ (see Lemma 3.3 in [1]).

Corollary 2.16. *Let $t \neq 0$ and c be generic. The module $J_{t,c}(\tau)$ is generated under $S\mathfrak{h}^*$ by homogeneous elements in degrees divisible by p . The images of such elements of degree mp in the quotient*

$$(J_{t,c}(\tau)/\mathfrak{h}^* J_{t,c}(\tau))_{mp} = J_{t,c}(\tau)_{mp}/\mathfrak{h}^* J_{t,c}(\tau)_{mp-1} \subseteq S^{mp}\mathfrak{h}^* \otimes \tau/\mathfrak{h}^* J_{t,c}(\tau)_{mp-1}$$

form a subrepresentation of $S^{mp}\mathfrak{h}^ \otimes \tau/\mathfrak{h}^* J_{t,c}(\tau)_{mp-1}$ whose composition factors are a sub-multiset of composition factors of $(S^m\mathfrak{h}^*)^p \otimes \tau/(\mathfrak{h}^* J'_{t,0}(\tau)_{mp-1} \cap (S^m\mathfrak{h}^*)^p \otimes \tau)$.*

Any such generator in degree mp is a singular vector in the quotient of $M_{t,c}(\tau)$ by the $S\mathfrak{h}^$ -submodule generated by all such generators from smaller degrees.*

Proof. For representations σ and σ' of G , let us write $\sigma \preceq \sigma'$ if the multiset of composition factors of σ is a subset of the multiset of composition factors of σ' . If σ and σ' are graded G representations, we write $\sigma \preceq \sigma'$ if $\sigma_i \preceq \sigma'_i$ for all i . If $\sigma \preceq \sigma'$ and $\sigma' \preceq \sigma$, then $[\sigma] = [\sigma']$ in the Grothendieck group.

Let us first prove:

$$J_{t,c}(\tau)/\mathfrak{h}^* J_{t,c}(\tau) \preceq J'_{t,0}(\tau)/\mathfrak{h}^* J'_{t,0}(\tau) \preceq (S\mathfrak{h}^*)^p \otimes \tau/(\mathfrak{h}^* J'_{t,0}(\tau) \cap (S\mathfrak{h}^*)^p \otimes \tau).$$

As $J_{t,c}(\tau)$ is a deformation of $J_{t,0}(\tau)'$, for any degree $i > 0$ we have, in the Grothendieck group:

$$\begin{aligned} [J_{t,0}(\tau)'_i] &= [J_{t,c}(\tau)_i] \\ [J_{t,0}(\tau)'_{i-1}] &= [J_{t,c}(\tau)_{i-1}] \\ \mathfrak{h}^* J_{t,0}(\tau)'_{i-1} &\preceq \mathfrak{h}^* J_{t,c}(\tau)'_{i-1} \end{aligned}$$

so

$$J_{t,c}(\tau)/\mathfrak{h}^* J_{t,c}(\tau) \preceq J'_{t,0}(\tau)/\mathfrak{h}^* J'_{t,0}(\tau).$$

The statement $J'_{t,0}(\tau)/\mathfrak{h}^* J'_{t,0}(\tau) \preceq (S\mathfrak{h}^*)^p \otimes \tau/(\mathfrak{h}^* J'_{t,0}(\tau) \cap (S\mathfrak{h}^*)^p \otimes \tau)$ follows from $J'_{t,0}(\tau)$ being generated under $S\mathfrak{h}^*$ by p -th powers.

The module $J_{t,c}(\tau)$ is generated under $S\mathfrak{h}^*$ by elements which have nonzero projection to $J_{t,c}(\tau)/\mathfrak{h}^* J_{t,c}(\tau)$. Because of the above sequence of \preceq , such elements only exist in degrees divisible by p , and their images in $J_{t,c}(\tau)/\mathfrak{h}^* J_{t,c}(\tau) \subset S\mathfrak{h}^* \otimes \tau/\mathfrak{h}^* J_{t,c}(\tau)$ form a group representation which is $\preceq (S\mathfrak{h}^*)^p \otimes \tau/(\mathfrak{h}^* J'_{t,0}(\tau) \cap (S\mathfrak{h}^*)^p \otimes \tau)$.

For every $v \in J_{t,c}(\tau)_{mp}$ and every $y \in \mathfrak{h}$, $D_y(v) \in J_{t,c}(\tau)_{mp-1}$. So, if v is not in $\mathfrak{h}^* J_{t,c}(\tau)_{mp-1}$, then its projection is a nonzero vector in $J_{t,c}(\tau)/\mathfrak{h}^* J_{t,c}(\tau)$ with a property that $D_y(v)$ is zero in $J_{t,c}(\tau)/\mathfrak{h}^* J_{t,c}(\tau)$, in other words a singular vector. \square

The reduced character $H(z)$ is computed as follows: if $J'_{t,0}(\tau)$ is generated by some collection of p -th powers $f_i(x_1^p, \dots, x_n^p) \otimes v_i$, then $H(z)$ is the character of the reduced module

$$R_{t,c}(\tau) = S\mathfrak{h}^* \otimes \tau / \langle f_i(x_1, \dots, x_n) \otimes v_i \rangle.$$

The reduced module $R_{t,c}(\tau)$ is a $\mathbb{k}[G] \ltimes S\mathfrak{h}^*$ -module, but does not have to be stable under Dunkl operators and might not be an $H_{t,c}(G, \mathfrak{h})$ -module.

For $G = GL_2(\mathbb{F}_p)$, the character of $S^{(p)}\mathfrak{h}^*$ is

$$\chi_{S^{(p)}\mathfrak{h}^*}(z) = \chi_{S\mathfrak{h}^*}(z) - 2\chi_{S\mathfrak{h}^*}(z)z^p + \chi_{S\mathfrak{h}^*}(z)z^{2p}$$

with the Hilbert series

$$\text{Hilb}_{S^{(p)}\mathfrak{h}^*}(z) = \left(\frac{1 - z^p}{1 - z} \right)^2.$$

3. THE GROUP $GL_2(\mathbb{F}_p)$

For the rest of the paper, we will be considering the rational Cherednik algebra associated to the group $GL_2(\mathbb{F}_p)$, for $\mathbb{F}_p \subseteq \mathbb{k}$ the finite field of p elements. This group has $(p^2 - 1)(p^2 - p)$ elements, and so p divides the order of the group and its category of representations is not semisimple. Let $\mathfrak{h}_{\mathbb{F}} = \mathbb{F}_p^2$ with the tautological representation, $\mathfrak{h} = \mathbb{k}^2$ with the tautological representation, y_1, y_2 be the tautological basis of \mathfrak{h} and x_1, x_2 the dual basis of \mathfrak{h}^* . For a group element $g \in GL_2(\mathbb{F}_p)$, the matrix of its action on \mathfrak{h} , written in y_1, y_2 , is g , and the matrix of its action on \mathfrak{h}^* , written in x_1, x_2 , is $(g^t)^{-1}$.

Recall that an element s of $GL_2(\mathbb{F}_p) \subseteq GL(\mathfrak{h})$ is called a *reflection* if $\text{rank}(1 - s)|_{\mathfrak{h}} = 1$. Reflections in $GL_2(\mathbb{F}_p)$ are elements that are conjugate, in $GL_2(\mathbb{F}_p)$, to one of the following elements:

$$d_{\lambda} = \begin{bmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{bmatrix}, \lambda \neq 1, \quad d_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

A reflection conjugate to one of the d_{λ} , $\lambda \neq 1$ is called a *semisimple reflection*, and a reflection conjugate to d_1 is called a *unipotent reflection*.

There are $p - 1$ conjugacy classes of reflections in $GL_2(\mathbb{F}_p)$, as for each $\lambda \in \mathbb{F}_p^{\times}$,

$$C_{\lambda} = \{gd_{\lambda}g^{-1} | g \in GL_2(\mathbb{F}_p)\}$$

is a conjugacy class of reflections. Let the value of the function $c : S \rightarrow \mathbb{k}$ on any element of C_{λ} be $c_{\lambda} \in \mathbb{k}$. Each semisimple conjugacy class C_{λ} , $\lambda \neq 1$ has $(p + 1)p$ elements, while C_1 contains $(p + 1)(p - 1)$ reflections.

Proposition 3.1. *The set of reflections can be parametrized as follows.*

- (1) *There exists a bijection between the set of reflections in $GL(\mathfrak{h}_{\mathbb{F}})$ and the set of all vectors $\alpha \otimes \alpha^{\vee} \neq 0$ in $\mathfrak{h}_{\mathbb{F}}^* \otimes \mathfrak{h}_{\mathbb{F}}$ such that $(\alpha, \alpha^{\vee}) \neq 1$. The reflection s corresponding to $\alpha \otimes \alpha^{\vee}$ acts:*

$$\begin{aligned} \text{on } \mathfrak{h}^* \text{ by } \quad s.x &= x - (\alpha^{\vee}, x)\alpha \\ \text{on } \mathfrak{h} \text{ by } \quad s.y &= y + \frac{(y, \alpha)}{1 - (\alpha, \alpha^{\vee})}\alpha^{\vee}. \end{aligned}$$

Such a reflection s is semisimple with nonunit eigenvalue $\lambda = 1 - (\alpha^{\vee}, \alpha)$ on \mathfrak{h}^ if $(\alpha, \alpha^{\vee}) \neq 0$, and unipotent if $(\alpha, \alpha^{\vee}) = 0$.*

- (2) *For $GL_2(\mathbb{F}_p)$, the conjugacy classes C_{λ} of reflections correspond, under the above bijection, to the following sets:*

$$\lambda \neq 1: \quad C_{\lambda} \leftrightarrow \left\{ \begin{bmatrix} 1 \\ b \end{bmatrix} \otimes \begin{bmatrix} 1 - \lambda - bd \\ d \end{bmatrix} \mid b, d \in \mathbb{F}_p \right\} \cup \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} a \\ 1 - \lambda \end{bmatrix} \mid a \in \mathbb{F}_p \right\}$$

$$C_1 \leftrightarrow \left\{ \begin{bmatrix} 1 \\ b \end{bmatrix} \otimes \begin{bmatrix} -bd \\ d \end{bmatrix} \mid b, d \in \mathbb{F}_p, d \neq 0 \right\} \cup \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} a \\ 0 \end{bmatrix} \mid a \in \mathbb{F}_p, a \neq 0 \right\}.$$

For a reflection s , call the vectors corresponding to it α_s and α_s^\vee . They are uniquely determined up to mutual normalization (meaning, we can replace α_s by $a \cdot \alpha_s$ if we replace α_s^\vee by $a^{-1} \cdot \alpha_s^\vee$). Note that α_s spans $\text{Im}(1-s)|_{\mathfrak{h}^*}$ and α_s^\vee spans $\text{Im}(1-s)|_{\mathfrak{h}}$, so this does not contradict the previously introduced notation of α_s as a basis vector in $\text{Im}(1-s)|_{\mathfrak{h}^*}$.

3.1. Invariants and characters of baby Verma modules. Proposition 2.13 states that the characters of baby Verma modules are easy to compute for groups G for which the algebra of invariants $(S\mathfrak{h}^*)^G$ is a polynomial algebra. Dickson in [3] showed that all $GL_n(\mathbb{F}_p)$ are such groups, and constructed the invariants explicitly. We recall the construction for $GL_2(\mathbb{F}_p)$.

For (n, m) an ordered pair of nonnegative integers, define $[n, m] \in S\mathfrak{h}^*$ to be the determinant

$$[n, m] = \det \begin{bmatrix} x_1^{p^n} & x_2^{p^n} \\ x_1^{p^m} & x_2^{p^m} \end{bmatrix}.$$

When acting on \mathfrak{h}^* by $g \in GL_2(\mathbb{F}_p)$, this element transforms as

$$g \cdot [n, m] = (\det g)^{-1} [n, m].$$

From this it follows that for $L = [1, 0]$,

$$Q_0 = L^{p-1} = [1, 0]^{p-1} = (x_1^p x_2 - x_1 x_2^p)^{p-1} = \sum_{i=0}^{p-1} x_1^{(p-1)(p-i)} x_2^{(p-1)(i+1)}$$

$$Q_1 = \frac{[2, 0]}{L} = \frac{[2, 0]}{[1, 0]} = \frac{x_1^{p^2} x_2 - x_1 x_2^{p^2}}{x_1^p x_2 - x_1 x_2^p} = \sum_{i=0}^p x_1^{(p-1)(p-i)} x_2^{(p-1)i}$$

are invariants. The main theorem of [3], applied to $GL_2(\mathbb{F}_p)$, is:

Proposition 3.2. *The ring of invariants $(S\mathfrak{h}^*)^{GL_2(\mathbb{F}_p)}$ is a polynomial ring generated by its algebraically independent elements Q_0 and Q_1 .*

Finally, as $\deg Q_0 = p^2 - 1$ and $\deg Q_1 = p^2 - p$, we get the character formulas for baby Verma modules for $GL_2(\mathbb{F}_p)$:

$$\chi_{N_{0,c}(\tau)}(z) = \chi_{M_{0,c}(\tau)}(z)(1 - z^{(p^2-1)})(1 - z^{(p^2-p)}),$$

$$\text{Hilb}_{N_{0,c}(\tau)}(z) = \dim(\tau) \frac{(1 - z^{(p^2-1)})(1 - z^{(p^2-p)})}{(1 - z)^2}.$$

$$\chi_{N_{1,c}(\tau)}(z) = \chi_{M_{1,c}(\tau)}(z)(1 - z^{p(p^2-1)})(1 - z^{p(p^2-p)}),$$

$$\text{Hilb}_{N_{1,c}(\tau)}(z) = \dim(\tau) \frac{(1 - z^{p(p^2-1)})(1 - z^{p(p^2-p)})}{(1 - z)^2}.$$

3.2. Representations of $GL_2(\mathbb{F}_p)$. Most of the results from this section can be found in [5], or proved directly.

Proposition 3.3. *All irreducible representations of $GL_2(\mathbb{F}_p)$ over \mathbb{k} are of the form*

$$S^i \mathfrak{h} \otimes (\det)^j,$$

for $i = 0, 1, \dots, p-1$, $j = 0, \dots, p-2$.

Proof. From the paper of Steinberg [7] it follows that all the irreducible representations of $SL_2(\mathbb{F}_p)$ are of the form $S^i \mathfrak{h}$. Any irreducible representation of $GL_2(\mathbb{F}_p)$ stays irreducible when restricted to $SL_2(\mathbb{F}_p)$. The group $GL_2(\mathbb{F}_p)$ is generated by $SL_2(\mathbb{F}_p)$ and the subgroup of elements of the form $\begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}$, which act by a character, say $\begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} \mapsto \lambda^j$ on the one-dimensional subrepresentation y_1^i . So, any irreducible representation of $GL_2(\mathbb{F}_p)$ is of the form $S^i \mathfrak{h} \otimes (\det)^j$. \square

Lemma 3.4. *As representations of $GL_2(\mathbb{F}_p)$,*

$$\mathfrak{h}^* \cong \mathfrak{h} \otimes (\det)^{-1},$$

and the isomorphism is $x_1 \mapsto -y_2$, $x_2 \mapsto y_1$.

Proof. The matrix $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{F}_p)$ acts on \mathfrak{h} , in basis y_1, y_2 , in a tautological way as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. It acts on \mathfrak{h}^* , in a basis x_1, x_2 , as

$$(g^{-1})^t = \frac{1}{\det} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} = \frac{1}{\det} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

\square

Next, we will need to know how the higher symmetric powers and tensor products of such representations decompose into irreducible components.

Proposition 3.5. *For any $i, j > 0$, there is a short exact sequence of $GL_2(\mathbb{F}_p)$ -representations*

$$0 \rightarrow S^{i-1} \mathfrak{h} \otimes S^{j-1} \mathfrak{h} \otimes \det \rightarrow S^i \mathfrak{h} \otimes S^j \mathfrak{h} \rightarrow S^{i+j} \mathfrak{h} \rightarrow 0.$$

The first map is $f \otimes g \mapsto (y_1 \otimes y_2 - y_2 \otimes y_1) \cdot f \otimes g$, and the second map is $f \otimes g \mapsto f \cdot g$.

Proof. One can see directly that both maps are indeed $GL_2(\mathbb{F}_p)$ -representation maps and that they compose to 0. Multiplication of factors $S^i \mathfrak{h} \otimes S^j \mathfrak{h} \rightarrow S^{i+j} \mathfrak{h} \rightarrow 0$ is surjective, multiplication by $(y_1 \otimes y_2 - y_2 \otimes y_1)$ is injective, and the dimensions agree: $i \cdot j + (i+j+1) = (i+1)(j+1)$, so this is really a short exact sequence of group representations. \square

Proposition 3.6. *Let $0 \leq j < p$, and $n \geq 0$. There is a short exact sequence of $GL_2(\mathbb{F}_p)$ -representations:*

$$0 \rightarrow S^j \mathfrak{h} \otimes S^n \mathfrak{h} \rightarrow S^{j+pn} \mathfrak{h} \rightarrow S^{p-j-2} \mathfrak{h} \otimes S^{n-1} \mathfrak{h} \otimes \det^{j+1} \rightarrow 0.$$

Here we use the convention $S^i \mathfrak{h} = 0$ if $i < 0$. The first map is

$$\alpha(y_1^a y_2^b \otimes y_1^c y_2^d) = y_1^{a+cp} y_2^{b+dp}$$

and the second, for $0 \leq a, b < p$, is

$$\beta(y_1^{a+pc} y_2^{b+pd}) = \begin{cases} \binom{a}{j+1} \cdot y_1^{a-j-1} y_2^{b-j-1} \otimes y_1^c y_2^d, & a+b = p+j \\ 0, & \text{otherwise} \end{cases}$$

Proof. Both α and β send monomials to monomials. Every monomial in $S^{j+pn}\mathfrak{h}$ can be written as $y_1^{a+cp} y_2^{b+dp}$ with $0 \leq a, b < p$. Then either $a+b = j$ and the monomial is in the image of α and the kernel of β , or $a+b = j+p$, so $y_1^{a+cp} y_2^{b+dp}$ is not in the image of α and $\beta(y_1^{a+cp} y_2^{b+dp}) \neq 0$. It is clear that α is injective and β surjective, so the sequence is really a short exact sequence of vector spaces.

It remains to see that both maps commute with $GL_2(\mathbb{F}_p)$ -action. The map α can be written as a composition of raising all monomials in the second tensor factor to the p -th power and multiplication of tensor factors, both of which are $GL_2(\mathbb{F}_p)$ -maps. To see that β is a $GL_2(\mathbb{F}_p)$ -map, let $y_1^{a+cp} y_2^{b+dp} \in S^{j+pn}\mathfrak{h}$, with $0 \leq a, b < n$, and assume first that $a+b = j, c+d = n$. Then $\beta(y_1^{a+cp} y_2^{b+dp}) = 0$, and for any $g \in GL_2(\mathbb{F}_p)$, $\beta(g \cdot (y_1^{a+cp} y_2^{b+dp})) = 0$.

Next, consider $y_1^{a+cp} y_2^{b+dp} \in S^{j+pn}\mathfrak{h}$, with $0 \leq a, b < n$, $a+b = p+j, c+d = n-1$, and let us show that applying β to this element commutes with the group action. For this it is enough to see that β commutes with $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, e \in \mathbb{F}_p$, which generate $GL_2(\mathbb{F}_p)$.

Using $\det A = -1$, we get

$$\begin{aligned} \beta(A \cdot (y_1^{a+cp} y_2^{b+dp})) &= \beta(y_1^{b+dp} y_2^{a+cp}) \\ &= \binom{b}{j+1} y_1^{b-j-1} y_2^{a-j-1} \otimes y_1^d y_2^c \\ A \cdot (\beta(y_1^{a+cp} y_2^{b+dp})) &= A \cdot \left(\binom{a}{j+1} y_1^{a-j-1} y_2^{b-j-1} \otimes y_1^c y_2^d \right) \\ &= (-1)^{j+1} \binom{a}{j+1} y_1^{b-j-1} y_2^{a-j-1} \otimes y_1^d y_2^c. \end{aligned}$$

However, these are the same because

$$(-1)^{j+1} \binom{a}{j+1} = \binom{j-a}{j+1} = \binom{b}{j+1}$$

so $\beta A = A\beta$.

Next, using $\det B = e$, we get

$$\beta(B \cdot (y_1^{a+cp} y_2^{b+dp})) = \binom{a}{j+1} e^{a+c} y_1^{a-j-1} y_2^{b-j-1} \otimes y_1^c y_2^d = B \cdot (\beta(y_1^{a+cp} y_2^{b+dp})).$$

Finally,

$$\begin{aligned}
\beta(C.(y_1^{a+cp}y_2^{b+dp})) &= \beta\left(\sum_{i=0}^b \sum_{l=0}^d \binom{b}{i} \binom{d}{l} y_1^{a+i} y_2^{b-i} y_1^{(c+l)p} y_2^{(d-l)p}\right) \\
&= \sum_{i=0}^{b-j-1} \sum_{l=0}^d \binom{b}{i} \binom{d}{l} \binom{a+i}{j+1} y_1^{a+i-j-1} y_2^{b-i-j-1} \otimes y_1^{c+l} y_2^{d-l} \\
C.(\beta(y_1^{a+cp}y_2^{b+dp})) &= C.\left(\binom{a}{j+1} y_1^{a-j-1} y_2^{b-j-1} \otimes y_1^c y_2^d\right) \\
&= \sum_{i=0}^{b-j-1} \sum_{l=0}^d \binom{b-j-1}{i} \binom{d}{l} \binom{a}{j+1} y_1^{a+i-j-1} y_2^{b-i-j-1} \otimes y_1^{c+l} y_2^{d-l}
\end{aligned}$$

So, the claim that $\beta C = C\beta$ is equivalent to showing that

$$\binom{b}{i} \binom{a+i}{j+1} = \binom{b-j-1}{i} \binom{a}{j+1}.$$

Using $a+b=p+j$ and $\binom{A}{i} = (-1)^i \binom{-A+i-1}{i}$, this is equivalent to

$$(-1)^i \binom{a+i-j-1}{i} \binom{a+i}{j+1} = (-1)^i \binom{a+i}{i} \binom{a}{j+1},$$

which is true because both left and right hand side are equal to

$$(-1)^i \frac{(a+i)(a+i-1)\dots(a-j)}{i!(j+1)!}.$$

□

3.3. A lemma about finite fields. We will finish this section with a lemma which is useful in computations.

Lemma 3.7. *Let $f \in \mathbb{k}[x_1, x_2, \dots, x_n]$ be a polynomial in n variables, for which there exists a variable x_i such that*

$$\deg_{x_i}(f) < p-1.$$

In particular, this is satisfied by all f such that $\deg(f) < n(p-1)$. Then

$$\sum_{x_1, \dots, x_n \in \mathbb{F}_p} f(x_1, \dots, x_n) = 0.$$

4. CATEGORY \mathcal{O} FOR THE RATIONAL CHEREDNIK ALGEBRA $H_{t,c}(GL_2(\mathbb{F}_p), \mathfrak{h})$

The main theorem of the paper is the following:

Theorem 4.1. *Up to a grading shift, any irreducible representations in category \mathcal{O} for the rational Cherednik algebra $H_{t,c}(GL_2(\mathbb{F}_p), \mathfrak{h})$ is isomorphic to $L_{t,c}(S^i \mathfrak{h} \otimes (\det)^j)$ for some $0 \leq i \leq p-1$ and $0 \leq j \leq p-2$. The characters and Hilbert series of these representations are as follows.*

For $t = 0$:

- If $0 \leq i \leq p-3$,

$$\chi_{L_{0,c}(S^i \mathfrak{h} \otimes (\det)^j)}(z) = [S^i \mathfrak{h} \otimes (\det)^j],$$

$$\text{Hilb}_{L_{0,c}(S^i \mathfrak{h} \otimes (\det)^j)}(z) = i+1.$$

- If $i = p - 2$

$$\chi_{L_{0,c}(S^i \mathfrak{h} \otimes (\det)^j)}(z) = [S^{p-2} \mathfrak{h} \otimes (\det)^j] + [S^{p-1} \mathfrak{h} \otimes (\det)^{j-1}]z + [S^{p-2} \mathfrak{h} \otimes (\det)^{j-1}]z^2,$$

$$\text{Hilb}_{L_{0,c}(S^{p-2} \mathfrak{h} \otimes (\det)^j)}(z) = (p-1) + pz + (p-1)z^2.$$

- If $i = p - 1$

$$\chi_{L_{0,c}(S^{p-1} \mathfrak{h} \otimes (\det)^j)}(z) = \chi_{M_{0,c}(S^{p-1} \mathfrak{h} \otimes (\det)^j)}(z)(1 - z^{p-1})(1 - z^{p^2-1}),$$

$$\text{Hilb}_{L_{0,c}(S^{p-1} \mathfrak{h} \otimes (\det)^j)}(z) = p \frac{(1 - z^{p-1})(1 - z^{p^2-1})}{(1 - z)^2}.$$

For $t = 1$,

$$\chi_{L_{1,c}(\tau)}(z) = H_{L_{1,c}(\tau)}(z^p) \chi_{S^{(p)} \mathfrak{h}^*}(z),$$

$$\text{Hilb}_{L_{1,c}(\tau)}(z) = h_{L_{1,c}(\tau)}(z^p) \left(\frac{1 - z^p}{1 - z} \right)^2,$$

where

$$\chi_{S^{(p)} \mathfrak{h}^*}(z) = \chi_{S \mathfrak{h}^*}(z) - 2\chi_{S \mathfrak{h}^*}(z)z^p + \chi_{S \mathfrak{h}^*}(z)z^{2p}$$

and the reduced character and Hilbert series is:

- If $0 \leq i \leq p - 3$,

$$H_{L_{1,c}(S^i \mathfrak{h} \otimes (\det)^j)}(z) = [S^i \mathfrak{h} \otimes (\det)^j],$$

$$h_{L_{1,c}(S^i \mathfrak{h} \otimes (\det)^j)}(z) = i + 1.$$

- If $i = p - 2$

$$H_{L_{1,c}(S^{p-2} \mathfrak{h} \otimes (\det)^j)}(z) = [S^{p-2} \mathfrak{h} \otimes (\det)^j] + [S^{p-1} \mathfrak{h} \otimes (\det)^{j-1}]z + [S^{p-2} \mathfrak{h} \otimes (\det)^{j-1}]z^2,$$

$$h_{L_{1,c}(S^{p-2} \mathfrak{h} \otimes (\det)^j)}(z) = (p-1) + pz + (p-1)z^2.$$

- If $i = p - 1$

$$H_{L_{1,c}(S^{p-1} \mathfrak{h} \otimes (\det)^j)}(z) = \chi_{M_{0,c}(S^{p-1} \mathfrak{h} \otimes (\det)^j)}(z)(1 - z^{p-1})(1 - z^{p^2-1}),$$

$$h_{L_{1,c}(S^{p-1} \mathfrak{h} \otimes (\det)^j)}(z) = p \frac{(1 - z^{p-1})(1 - z^{p^2-1})}{(1 - z)^2}.$$

Proof. Lemma 4.7 shows that all the formulas for the characters of $L_{t,c}(S^i \mathfrak{h} \otimes (\det)^j)$ follow from the ones for $j = 0$. Those are proved for $0 \leq i < p - 2$ in Propositions 5.1 (for $t = 0$) and 5.2 (for $t = 1$); for $i = p - 2$ in Propositions 6.1 (for $t = 0$) and 6.5 (for $t = 1$), and for $i = p - 1$ in Propositions 7.5 and 7.6 (for $t = 0$) Propositions 7.8 and 7.9 (for $t = 1$). \square

Remark 4.2. For $G = GL_2(\mathbb{F}_p)$, one can notice from the previous theorem that the reduced character for $L_{1,c}(\tau)$ is equal to the character of $L_{0,c}(\tau)$ for all τ . As the analogous statement is always true for baby Verma modules, one might be tempted to conjecture that it holds in general. This is however not true: a counterexample is $G = SL_2(\mathbb{F}_3)$, $\tau = \text{triv}$.

4.1. **Blocks.** Consider the following element of $H_{t,c}(GL_2(\mathbb{F}_p), \mathfrak{h})$:

$$\mathbf{h} = \sum_{i=1,2} x_i y_i - \sum_{s \in S} c_s s.$$

For every conjugacy class C of reflections, $\sum_{s \in C} s$ is central in the group algebra of $GL_2(\mathbb{F}_p)$, so it acts as a constant on every irreducible representation. Now consider the lowest weight subspace $\tau \cong M_{t,c}(\tau)_0 \subseteq M_{t,c}(\tau)$: the sum of $x_i y_i$ acts on it as 0, so \mathbf{h} acts on it as a constant $-\sum_{s \in S} c_s s|_\tau$. Call this constant $h_c(\tau)$ (it does not depend on t). Direct computation shows that

$$[\mathbf{h}, x] = tx, [\mathbf{h}, y] = -ty, [\mathbf{h}, g] = 0$$

for $x \in \mathfrak{h}^*, y \in \mathfrak{h}, g \in G$. This implies the following lemma:

Lemma 4.3. *For $t = 0$, the element \mathbf{h} is central and acts by a constant $h_c(\tau)$ on $M_{0,c}(\tau)$. For $t = 1$, \mathbf{h} acts by $h_c(\tau)$ on $M_{1,c}(\tau)_0$ and by $h_c(\tau) + i \in \mathbb{k}$ on $M_{1,c}(\tau)_i$.*

A consequence of this is:

Lemma 4.4. *For $t = 0, 1$, if $L_{t,c}(\sigma)[m]$ is a composition factor of $M_{t,c}(\tau)$ or $N_{t,c}(\tau)$, then*

$$h_c(\sigma) - h_c(\tau) = t \cdot m \in \mathbb{Z}/p\mathbb{Z}.$$

In particular, this happens if M is any quotient of $M_{t,c}(\tau)$, and $\sigma \subseteq M_m$ is a $GL_2(\mathbb{F}_p)$ subrepresentation consisting of singular vectors.

For $t = 1$ and generic c , the Hilbert series is of the form $(\frac{1-z^p}{1-z})^2 h(z^p)$, so the only composition factor in $M_{1,c}(\tau)$ and $N_{1,c}(\tau)$ are of the form $L_{1,c}(\sigma)[mp]$. Hence, for $t = 0$ or for $t = 1$ and generic c , the above condition reduces to

$$h_c(\sigma) = h_c(\tau),$$

and separates representations into blocks.

The constants $h_c(\tau)$ are easy to calculate directly.

Lemma 4.5. *For $GL_2(\mathbb{F}_p)$ and conjugacy class C_λ of reflections in it, the action of central elements $\sum_{s \in C_\lambda} s$ on symmetric powers of the reflection representation is:*

$$\text{For } \lambda \neq 1, \quad \sum_{s \in C_\lambda} s|_{S^i \mathfrak{h}} = \begin{cases} 0, & i < p-1 \\ 1, & i = p-1 \end{cases}$$

$$\text{For } \lambda = 1, \quad \sum_{s \in C_1} s|_{S^i \mathfrak{h}} = \begin{cases} -1, & i < p-1 \\ 0, & i = p-1 \end{cases}$$

So, for $\tau = S^i \mathfrak{h} \otimes \det^j$, the action of \mathbf{h} on the lowest weight $\tau \subseteq M_{t,c}(\tau)$ is by the constant

$$h_c(\tau) = \begin{cases} c_1, & i < p-1 \\ -\sum_{\lambda \neq 0,1} \lambda^j c_\lambda, & i = p-1 \end{cases}$$

Proof. We use the parametrization of conjugacy classes from Proposition 3.1 .

As $\sum_{s \in C_\lambda} s$ is central in the group algebra and acts on $S^i \mathfrak{h}$ as a constant, it is enough to compute $\sum_{s \in C_\lambda} s \cdot y_1^i$. As we are computing it, we may disregard all terms of the type $y_1^{i-j} y_2^j$ for $j > 0$, as we know these sum up to zero. We use Lemma 3.7 several times.

For $\lambda \neq 1$, the action of $\sum_{s \in C_\lambda} s$ on y_1^i is by a constant:

$$\begin{aligned} & \sum_{b,d \in \mathbb{F}_p} \left(1 + \frac{1}{\lambda} \cdot 1 \cdot (1 - \lambda - bd)\right)^i + \sum_{a \in \mathbb{F}_p} \left(1 + \frac{1}{\lambda} \cdot 0\right)^i = \\ &= \frac{1}{\lambda^i} \sum_{b,d \in \mathbb{F}_p} (1 - bd)^i = \frac{-1}{\lambda^i} \sum_{m \in \mathbb{F}_p} m^i = - \sum_{m \in \mathbb{F}_p} m^i = \begin{cases} 0, & i < p-1 \\ 1, & i = p-1 \end{cases} \end{aligned}$$

For $\lambda = 1$, a similar computation yields:

$$\sum_{\substack{b,d \in \mathbb{F}_p \\ d \neq 0}} (1 - bd)^i + \sum_{\substack{a \in \mathbb{F}_p \\ a \neq 0}} 1^i = (p-1) \left(\sum_{m \in \mathbb{F}_p} m^i + 1 \right) = \begin{cases} -1, & i < p-1 \\ 0, & i = p-1. \end{cases}$$

The formulas for the action of \mathbf{h} on irreducible representations $S^i \mathfrak{h} \otimes \det^j$ is now computed from this directly. \square

Corollary 4.6. *For generic c , the representations of the form $L_{t,c}(S^{p-1} \mathfrak{h} \otimes \det^j)$ form blocks of size one, meaning that the only irreducible representations that appear as composition factors in any representation with lowest weight $S^{p-1} \mathfrak{h} \otimes \det^j$ are isomorphic, up to grading shifts, to $L_{t,c}(S^{p-1} \mathfrak{h} \otimes \det^j)$.*

4.2. Dependence of the character of $L_{t,c}(S^i \mathfrak{h} \otimes (\det)^j)$ on j . We start the character computations with a reduction which allows us to only consider the case $j = 0$. The following lemma is true for any finite reflection group G , realized as a subgroup of $GL(\mathfrak{h})$.

Lemma 4.7. *The algebras $H_{t,c}(G, \mathfrak{h})$ and $H_{t,c \cdot \det}(G, \mathfrak{h})$ are isomorphic. Therefore, for generic c and any irreducible representation τ of G , the Hilbert series of $L_{t,c}(\tau \otimes (\det)^j)$ does not depend on j , and their characters are related by*

$$\chi_{L_{t,c}(\tau \otimes \det)} = \chi_{L_{t,c}(\tau)} \cdot [\det].$$

Proof. The isomorphism $H_{t,c}(G, \mathfrak{h}) \rightarrow H_{t,c \cdot \det}(G, \mathfrak{h})$ is defined on generators of the rational Cherednik algebra to be the identity on \mathfrak{h} and \mathfrak{h}^* , and to send $g \in G$ to $(\det g) \cdot g$. Twisting by this isomorphism makes a representation $L_{t,c \cdot \det}(\tau)$ of $H_{t,c \cdot \det}(G, \mathfrak{h})$ into a representation $L_{t,c}(\tau \otimes \det)$ of $H_{t,c}(G, \mathfrak{h})$. So, picking c is such that both c and $c \cdot \det$ are generic parameters, the Hilbert series of $L_{t,c}(\tau)$ and $L_{t,c}(\tau \otimes \det)$ are the same, and that their characters satisfy

$$\chi_{L_{t,c}(\tau \otimes \det)} = \chi_{L_{t,c \cdot \det}(\tau)} \cdot [\det] = \chi_{L_{t,c}(\tau)} \cdot [\det]$$

(here, multiplication is in the Grothendieck ring and corresponds to taking tensor products of representations). \square

Note this is false for special values of parameter c ; more specifically, it shows that c is special for $\tau \otimes \det$ if and only if $c \cdot \det$ is special for τ .

Because of this lemma, for $G = GL_2(\mathbb{F}_p)$ it is enough to calculate the characters of $L_{t,c}(S^i \mathfrak{h})$ for generic c .

5. CHARACTERS OF $L_{t,c}(S^i\mathfrak{h})$ FOR $i = 0 \dots p-3$

5.1. Characters of $L_{t,c}(S^i\mathfrak{h})$ for $i = 0 \dots p-3$ and $t = 0$.

Proposition 5.1. *For $i = 0, \dots, p-3$, $t = 0$ and all c , the space $M_{0,c}(S^i\mathfrak{h})_1$ consists of singular vectors. So, the character of $L_{0,c}(S^i\mathfrak{h})$ is*

$$\chi_{L_{0,c}(S^i\mathfrak{h})}(z) = [S^i\mathfrak{h}], \quad \text{Hilb}_{L_{0,c}(S^i\mathfrak{h})}(z) = i + 1.$$

Proof. The space $M_{0,c}(S^i\mathfrak{h})_1$ is isomorphic to $\mathfrak{h}^* \otimes S^i\mathfrak{h}$ as a $GL_2(\mathbb{F}_p)$ -representation. To show that it consists of singular vectors, we will show that for any $x \in \mathfrak{h}^*$, any $y \in \mathfrak{h}$, and any $f \in S^i\mathfrak{h}$,

$$D_y(x \otimes f) = t(y, x) - \sum_{s \in S} c_s(y, \alpha_s) \frac{(1-s).x}{\alpha_s} \otimes s.f$$

is zero. As $t = 0$ and we are claiming this holds for all c , it is equivalent to showing that, for any conjugacy class C_λ of reflections,

$$\sum_{s \in C_\lambda} (y, \alpha_s) \frac{(1-s).x}{\alpha_s} \otimes s.f$$

is zero.

Using Proposition 3.1 (1), this sum is equal to

$$\sum_{\substack{\alpha \otimes \alpha^\vee \neq 0 \\ (\alpha, \alpha^\vee) = 1-\lambda}} (y, \alpha)(x, \alpha^\vee) \otimes s.f.$$

We now use Proposition 3.1 (2), which parametrizes all $\alpha \otimes \alpha^\vee$ such that $(\alpha, \alpha^\vee) = 1 - \lambda$ as nonzero vectors in $\left\{ \begin{bmatrix} 1 \\ b \end{bmatrix} \otimes \begin{bmatrix} 1-\lambda-bd \\ d \end{bmatrix} \mid b, d \in \mathbb{F}_p \right\} \cup \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} a \\ 1-\lambda \end{bmatrix} \mid a \in \mathbb{F}_p \right\}$. We write the above sum as a sum over $a, b, d \in \mathbb{F}_p$ which produce nonzero elements of this set. First note that if α or α^\vee are zero, they do not contribute to the sum, so we can sum over all $a, b, d \in \mathbb{F}_p$. Next, note that $s.f$ is a polynomial of degree i in a and in d , so the summand $(y, \alpha)(x, \alpha^\vee) \otimes s.f$ is a polynomial in degree $1+i \leq p-2 < p-1$ in a and in d . By Lemma 3.7, this means the sum is zero, as claimed. \square

5.2. Characters of $L_{t,c}(S^i\mathfrak{h})$ for $i = 0 \dots p-3$ and $t = 1$. A very similar computation gives the analogous answer in case $t = 1$.

Proposition 5.2. *For $i = 0, \dots, p-3$, $t = 1$ and all c , all the vectors of the form $x^p \otimes v \in S^p\mathfrak{h}^* \otimes S^i\mathfrak{h}$ are singular. For generic c these vectors generate $J_{1,c}(S^i\mathfrak{h})$, and the character of the irreducible module $L_{1,c}(S^i\mathfrak{h})$ is*

$$\chi_{L_{1,c}(S^i\mathfrak{h})}(z) = \chi_{S^{(p)}\mathfrak{h}^*}(z) \cdot [S^i\mathfrak{h}],$$

its Hilbert series is

$$\text{Hilb}_{L_{1,c}(S^i\mathfrak{h})}(z) = (i+1) \left(\frac{1-z^p}{1-z} \right)^2,$$

and its reduced character and Hilbert series are

$$H(z) = [S^i\mathfrak{h}], \quad h(z) = i + 1.$$

Proof. The proof is very similar to the proof of the previous proposition. To show that all vectors of the form $x^p \otimes v \in S^p \mathfrak{h}^* \otimes S^i \mathfrak{h} \cong M_{1,c}(S^i \mathfrak{h})_p$ are singular, we need to show that the

$$D_y(x^p \otimes f) = \partial_y x^p - \sum_{s \in S} c_s(y, \alpha_s) \frac{(x, \alpha_s^\vee)^p \alpha_s^p}{\alpha_s} \otimes s.f = - \sum_{\lambda} c_{\lambda} \sum_{\alpha} (y, \alpha) \alpha^{p-1} \sum_{\alpha^\vee} (x, \alpha^\vee)^p \otimes (s.f)$$

is zero. Again use Proposition 3.1 to write this as a sum over all $a, b, d \in \mathbb{F}_p$ parametrizing $\alpha \otimes \alpha^\vee$. We may assume that $x \in \mathfrak{h}_{\mathbb{F}}^* \subseteq \mathfrak{h}^*$ by picking a convenient basis; the claim will then be true for \mathbb{k} -linear combinations of such x as well. For $x \in \mathfrak{h}_{\mathbb{F}}^*$, (x, α^\vee) is in $\mathbb{F}_p \subseteq \mathbb{k}$, and $(x, \alpha^\vee)^p = (x, \alpha^\vee)$. Using this, the inner sum over α^\vee again becomes a sum over all $d \in \mathbb{F}_p$ or over all $a \in \mathbb{F}_p$ of a polynomial $(x, \alpha^\vee) \otimes (s.f)$ of degree $1 + i < p - 1$ in d or a , so the sum is zero by Lemma 3.7.

To see these vectors generate $J_{1,c}(S^i \mathfrak{h})$ at generic c , we use Proposition 2.14, by which the character of $L_{1,c}(S^i \mathfrak{h})$ is of the form

$$\chi_{L_{1,c}(\tau)}(z) = \chi_{S^{(p)} \mathfrak{h}^*}(z) H(z^p).$$

The character of the quotient of $M_{1,c}(S^i \mathfrak{h})$ by the singular vectors found in this lemma is

$$\chi_{L_{1,c}(\tau)}(z) = \chi_{S^{(p)} \mathfrak{h}^*}(z) [S^i \mathfrak{h}],$$

and so the graded $GL_2(\mathbb{F}_p)$ representation with the character $H(z)$ is a quotient of the irreducible representation concentrated in one degree $[S^i \mathfrak{h}]$. As it is nonzero, there is no other choice then $H(z) = [S^i \mathfrak{h}]$, so the character of $L_{1,c}(S^i \mathfrak{h})$ is as claimed, and the maximal graded submodule $J_{1,c}(S^i \mathfrak{h})$ is generated by singular vectors of the form $x^p \otimes f$. \square

Note this proposition says nothing about the character at special values of c ; we can only conclude that for some special values of c , the modules $L_{1,c}(S^i \mathfrak{h})$ are smaller than the above described modules for generic c . The vectors $x^p \otimes f$ are still singular, but for particular values of c the character does not have to be of the form $\chi_{S^{(p)} \mathfrak{h}^*}(z) H(z^p)$, so there could be other singular vectors in degrees $1, 2, \dots, p-1$.

In the appropriately chosen Grothendieck ring, for $d_1 = p^2 - p$, $d_2 = p^2 - 1$, and using

$$\chi_{N_{1,c}(\tau)} = \chi_{M_{1,c}(\tau)}(1 - z^{pd_1})(1 - z^{pd_2}),$$

we have

$$L_{1,c}(S^i \mathfrak{h}) = M_{1,c}(S^i \mathfrak{h}) - M_{1,c}(\mathfrak{h}^* \otimes S^i \mathfrak{h})[p] + M_{1,c}(\det \otimes S^i \mathfrak{h})[2p]$$

and

$$\chi_{L_{1,c}(S^i \mathfrak{h})} = \frac{\chi_{N_{1,c}(S^i \mathfrak{h})}(z) - \chi_{N_{1,c}(\mathfrak{h}^* \otimes S^i \mathfrak{h})}(z) z^p + \chi_{N_{1,c}(\det \otimes S^i \mathfrak{h})}(z) z^{2p}}{(1 - z^{pd_1})(1 - z^{pd_2})}.$$

6. CHARACTERS OF $L_{t,c}(S^i \mathfrak{h})$ FOR $i = p - 2$

6.1. Characters of $L_{t,c}(S^i \mathfrak{h})$ for $i = p - 2$ and $t = 0$.

Proposition 6.1. *The character of $L_{0,c}(S^{p-2} \mathfrak{h})$ is*

$$[S^{p-2} \mathfrak{h}] + [S^{p-1} \mathfrak{h} \otimes (\det)^{-1}]z + [S^{p-2} \mathfrak{h} \otimes (\det)^{-1}]z^2,$$

and its Hilbert series is

$$(p-1) + pz + (p-1)z^2.$$

Proof. We will prove this in a series of lemmas. Let us outline the proof here, and define several auxiliary modules, used only in this subsection.

The character of the Verma module $M_{0,c}(S^{p-2}\mathfrak{h})$ is

$$\chi_{M_{0,c}(S^{p-2}\mathfrak{h})}(z) = \sum_{j \geq 0} [S^j \mathfrak{h}^* \otimes S^{p-2}\mathfrak{h}] z^j.$$

Lemma 6.2 shows that the space of singular vectors in $M_{0,c}(S^{p-2}\mathfrak{h})_1$ is isomorphic to $S^{p-3}\mathfrak{h}$, and consequently that $J_{0,c}(S^{p-2}\mathfrak{h})_1 \cong S^{p-3}\mathfrak{h}$. We define M^1 to be the quotient of the Verma module $M_{0,c}(S^{p-2}\mathfrak{h})$ by the submodule generated by these vectors.

The character of M^1 begins as

$$\begin{aligned} \chi_{M^1}(z) = & [S^{p-2}\mathfrak{h}] + ([\mathfrak{h}^* \otimes S^{p-2}\mathfrak{h}] - [S^{p-3}\mathfrak{h}])z + ([S^2\mathfrak{h}^* \otimes S^{p-2}\mathfrak{h}] - [\mathfrak{h}^* \otimes S^{p-3}\mathfrak{h}])z^2 + \\ & + ([S^3\mathfrak{h}^* \otimes S^{p-2}\mathfrak{h}] - [S^2\mathfrak{h}^* \otimes S^{p-3}\mathfrak{h}])z^3 + \dots, \end{aligned}$$

which is, using Lemmas 3.5 and 3.6, equal to

$$\chi_{M^1}(z) = [S^{p-2}\mathfrak{h}] + [S^{p-1}\mathfrak{h} \otimes (\det)^{-1}]z + [S^p\mathfrak{h} \otimes (\det)^{-2}]z^2 + [S^{p+1}\mathfrak{h} \otimes (\det)^{-3}]z^3 + \dots$$

The module M^1 has the property that its zero-th and first graded piece are equal to those of the irreducible module $L_{0,c}(\tau)$.

However, M^1 is not irreducible. Lemma 6.3 shows that the space of singular vectors in $M_2^1 \cong S^p\mathfrak{h} \otimes (\det)^{-2}$ is isomorphic to $\mathfrak{h} \otimes (\det)^{-2}$. This subspace is thus also in $J_{0,c}(S^{p-2}\mathfrak{h})$. Define M^2 as the quotient of M^1 by the submodule generated by these vectors. M^2 is equal to $L_{0,c}(S^{p-2}\mathfrak{h})$ in graded pieces 0, 1 and 2, and its character begins as

$$\begin{aligned} \chi_{M^2}(z) = & [S^{p-2}\mathfrak{h}] + [S^{p-1}\mathfrak{h} \otimes (\det)^{-1}]z + ([S^p\mathfrak{h} \otimes (\det)^{-2}] - [\mathfrak{h} \otimes (\det)^{-2}])z^2 + \\ & + ([S^{p+1}\mathfrak{h} \otimes (\det)^{-3}] - [\mathfrak{h}^* \otimes \mathfrak{h} \otimes (\det)^{-2}])z^3 + \dots = \\ = & [S^{p-2}\mathfrak{h}] + [S^{p-1}\mathfrak{h} \otimes (\det)^{-1}]z + [S^{p-2}\mathfrak{h} \otimes (\det)^{-1}]z^2 + ([S^{p-3}\mathfrak{h} \otimes (\det)^{-1}]z^3 + \dots \end{aligned}$$

Finally, Lemma 6.4 shows that $M_3^2 \cong S^{p-3}\mathfrak{h} \otimes (\det)^{-1}$ is entirely made of singular vectors. From this it follows that the quotient of M^2 by this subspace, which is an $H_{0,c}(GL_2(\mathbb{F}_p), \mathfrak{h})$ -module with character

$$[S^{p-2}\mathfrak{h}] + [S^{p-1}\mathfrak{h} \otimes (\det)^{-1}]z + [S^{p-2}\mathfrak{h}(\det)^{-1}]z^2,$$

is irreducible and equal to $L_{0,c}(S^{p-2}\mathfrak{h})$.

This proves the proposition, modulo Lemmas 6.2, 6.3 and 6.4. □

Lemma 6.2. *The space of singular vectors in $M_{0,c}(S^{p-2}\mathfrak{h})_1 \cong \mathfrak{h}^* \otimes S^{p-2}\mathfrak{h}$ is isomorphic to $S^{p-3}\mathfrak{h}$ and consists of all vectors of the form*

$$x_1 \otimes y_1 f + x_2 \otimes y_2 f, \quad f \in S^{p-3}\mathfrak{h}.$$

Proof. As a $GL_2(\mathbb{F}_p)$ -representation, the first graded piece of the Verma module, $M_{0,c}(S^{p-2}\mathfrak{h})_1$ is isomorphic to $\mathfrak{h}^* \otimes S^{p-2}\mathfrak{h}$. By Lemma 3.4, this is isomorphic to $\mathfrak{h} \otimes S^{p-2}\mathfrak{h} \otimes (\det)^{-1}$, and by Lemma 3.5, it fits into a short exact sequence

$$0 \rightarrow S^{p-3}\mathfrak{h} \rightarrow \mathfrak{h} \otimes S^{p-2}\mathfrak{h} \otimes (\det)^{-1} \rightarrow S^{p-1}\mathfrak{h} \otimes (\det)^{-1}\mathfrak{h} \rightarrow 0.$$

The irreducible subrepresentation isomorphic to $S^{p-3}\mathfrak{h}$ includes into $\mathfrak{h}^* \otimes S^{p-2}\mathfrak{h}$ by $f \mapsto x_1 \otimes y_1 f + x_2 \otimes y_2 f$. Both this subrepresentation and the quotient are irreducible.

If a vector $v \in M_{0,c}(S^{p-2}\mathfrak{h})_1$ is contained in the kernel of B , which is the maximal proper graded submodule $J_{0,c}(S^{p-2}\mathfrak{h})$, then action on it by $y \in \mathfrak{h}$ produces an element of $J_{0,c}(S^{p-2}\mathfrak{h})_0$.

However, the form is nondegenerate in degree 0, and $L_{0,c}(S^{p-2}\mathfrak{h})_0 = M_{0,c}(S^{p-2}\mathfrak{h})_0$, so $y.v = 0$. In other words, such a vector is singular.

To show that $J_{0,c}(S^{p-2}\mathfrak{h})_1 \cong S^{p-3}\mathfrak{h}$, we are going to show that:

- (1) At least one nonzero vector from $S^{p-3}\mathfrak{h}$ is singular;
- (2) Not all vectors in $M_{0,c}(S^{p-2}\mathfrak{h})_1$ are singular.

The space of singular vectors is invariant under the group action, and both $S^{p-3}\mathfrak{h}$ and the quotient are irreducible, so this proves the claim.

First, let us show that the space $S^{p-3}\mathfrak{h}$ consist of singular vectors. The set of vectors $x_1 \otimes y_1 f + x_2 \otimes y_2 f$ is symmetric with respect to changing indices 1 and 2, so it is enough to show that

$$D_{y_1}(x_1 \otimes y_1 f + x_2 \otimes y_2 f)$$

is zero. We use parametrization of conjugacy classes from Proposition 3.1 and the definition of Dunkl operator D_{y_1} , and denote $\alpha_b = x_1 + b x_2$, and see that the coefficient of $-c_\lambda$ in $D_{y_1}(x_1 \otimes y_1 f + x_2 \otimes y_2 f)$ is

$$\begin{aligned} & \sum_{\alpha \otimes \alpha^\vee \in C_\lambda} (y_1, \alpha) \frac{1}{\alpha} ((x_1 - s.x_1) \otimes (s.y_1)(s.f) + (x_2 - s.x_2) \otimes (s.y_2)(s.f)) \\ &= \sum_{b,d} 1 \cdot \frac{1}{\alpha_b} ((1 - \lambda - bd)\alpha_b \otimes (s.y_1)(s.f) + d\alpha_b \otimes (s.y_2)(s.f)) \\ &= 1 \otimes \sum_{b,d} ((1 - \lambda - bd)s.y_1 + ds.y_2) s.f \\ &= 1 \otimes \sum_{b,d} \frac{1}{\lambda} ((1 - \lambda - bd)y_1 + dy_2) s.f. \end{aligned}$$

The sum is over all $b, d \in \mathbb{F}_p$ if $\lambda \neq 1$, and over all $b, d \in \mathbb{F}_p$ with $d \neq 0 \in \mathbb{F}_p$ if $\lambda = 1$. However, if $\lambda = 1$, then the $d = 0$ term does not contribute to the sum, so let us consider the sum to be over all $b, d \in \mathbb{F}_p$ in both cases. The term $s.f$ is a vector in $S^{p-3}\mathfrak{h}$ with coefficients polynomials in b, d whose degree in b and in d is less or equal to $p-3$. The overall expression is a sum over all $b, d \in \mathbb{F}_p$ of polynomials whose degree in each variable is $\leq p-2$, and it is thus zero by Lemma 3.7.

So, the subspace isomorphic to $S^{p-3}\mathfrak{h}$ indeed consists of singular vectors.

To see that the space of singular vectors in $M_{0,c}(S^{p-2}\mathfrak{h})_1$ is not the whole space, it is enough to find one vector which is not singular. For example, the above computation shows that $D_{y_1}(x_1 \otimes (y_1)^{p-2})$ has a coefficient of $-c_1 y_1^{p-2}$ equal to

$$- \sum_{b,d \in \mathbb{F}_p} bd(1 - bd)^{p-2} = - \sum_{b,d \in \mathbb{F}_p} \sum_{k=0}^{p-2} \binom{p-2}{k} (-1)^k (bd)^{k+1} = \sum_{b,d \in \mathbb{F}_p} (bd)^{p-1} = 1 \neq 0,$$

so $x_1 \otimes (y_1)^{p-2}$ is not singular. □

In the proof of Proposition 6.1 we defined M^1 as the quotient of $M_{0,c}(S^{p-2}\mathfrak{h})$ by the submodule generated by singular vectors $x_1 \otimes y_1 f + x_2 \otimes y_2 f$, $f \in S^{p-3}\mathfrak{h}$ from the previous lemma. It is explained in this proof that M^1 agrees with $L_{0,c}(S^{p-2}\mathfrak{h})$ in graded pieces 0 and 1, and that its character is

$$\chi_{M^1}(z) = [S^{p-2}\mathfrak{h}] + [S^{p-1}\mathfrak{h} \otimes (\det)^{-1}]z + [S^p\mathfrak{h} \otimes (\det)^{-2}]z^2 + [S^{p+1}\mathfrak{h} \otimes (\det)^{-3}]z^3 + \dots$$

Next, we find the subspace of M_2^1 which is in $\text{Ker } B$, and which is by the same argumentation as in the proof of Lemma 6.2 equal to the space of singular vectors in M_2^1 .

Lemma 6.3. *The space of singular vectors in M_2^1 is isomorphic to $\mathfrak{h} \otimes (\det)^{-2}$. The representatives of these vectors in the Verma module $M_{0,c}(S^{p-2}\mathfrak{h})$ are linear combinations of $x_2^2 \otimes y_1^{p-2}$ and $x_1^2 \otimes y_2^{p-2}$.*

Proof. The space $M_2^1 \cong S^p\mathfrak{h} \otimes (\det)^{-2}$ fits, by Lemma 3.6, into a short exact sequence of $GL_2(\mathbb{F}_p)$ representations

$$0 \rightarrow \mathfrak{h} \otimes (\det)^{-2} \rightarrow S^p\mathfrak{h} \otimes (\det)^{-2} \rightarrow S^{p-2}\mathfrak{h} \otimes \det \rightarrow 0.$$

We are claiming that the irreducible subrepresentation consists of singular vectors, but that the quotient is not in the kernel of B . Tracking through all the inclusions, quotient maps and isomorphisms in the previous lemma shows that $x_2^2 \otimes y_1^{p-2}$ and $x_1^2 \otimes y_2^{p-2} \in M_{0,c}(S^{p-2}\mathfrak{h})_2$ really map to the basis of $\mathfrak{h} \otimes (\det)^{-2}$ under the quotient map $M_{0,c}(S^{p-2}\mathfrak{h}) \rightarrow M_2^1$.

We use the following observation. For any rational Cherednik algebra module N , and any $y \in \mathfrak{h}$, $n \in N$, $g \in G$, the relations of the rational Cherednik algebra imply that $g \cdot (y \cdot n) = (g \cdot y) \cdot (g \cdot n)$, so the map $\mathfrak{h} \otimes N_i \rightarrow N_{i-1}$ given by $y \otimes n \mapsto y \cdot n$ is a map of $GL_2(\mathbb{F}_p)$ -representations. So, if N is a quotient of the Verma module and D_y the induced Dunkl operator on the submodule, then the map $y \otimes n \mapsto D_y(n)$ is a map of group representations.

In particular, applying the Dunkl operator is a homomorphism

$$\mathfrak{h} \otimes M_2^1 \rightarrow M_1^1.$$

Showing that $\mathfrak{h} \otimes (\det)^{-2} \subseteq M_2^1$ consists of singular vectors is equivalent to showing that the restriction of the above map to this space, which is

$$\mathfrak{h} \otimes \mathfrak{h} \otimes (\det)^{-2} \rightarrow M_1^1,$$

is zero. To do this, notice that the short exact sequence calculating the composition series of $\mathfrak{h} \otimes \mathfrak{h} \otimes (\det)^{-2}$ is

$$0 \rightarrow (\det)^{-1} \rightarrow \mathfrak{h} \otimes \mathfrak{h} \otimes (\det)^{-2} \rightarrow S^2\mathfrak{h} \otimes (\det)^{-2} \rightarrow 0,$$

while the target space of the homomorphism is the irreducible $M_1^1 \cong S^{p-1}\mathfrak{h} \otimes (\det)^{-1}$. By Schur's lemma, this homomorphism has to be zero, and thus $\mathfrak{h} \otimes (\det)^{-2}$ consists of singular vectors.

To see these are all singular vectors in M_2^1 , it is enough to find one nonsingular vector, because the quotient of M_2^1 by $\mathfrak{h} \otimes (\det)^{-2}$ is irreducible representation. Direct computation shows that $D_{y_1}x_1x_2 \otimes y_1^{p-2}$ has c_1 coefficient equal to $x_2 \otimes y_1^{p-2}$, which is nonzero in M^1 . \square

The next module to consider is M^2 , defined as the quotient of M^1 by the singular vectors from the previous lemma. The irreducible module $L_{0,c}(S^{p-2}\mathfrak{h})$ is a quotient of M^2 , and they agree in degrees 0, 1, 2. We proceed looking for singular vectors in M_3^2 , which turn out to be all of it.

Lemma 6.4. *All vectors in $M_3^2 \cong S^{p-3}\mathfrak{h} \otimes (\det)^{-1}$ are singular.*

Proof. We are going to prove the lemma in two steps: first, we show that the claim follows from showing that the image of $D_{y_1}(x_1^2x_2 \otimes y_1^{p-2})$ in M_2^2 is zero, and then showing this is true.

First, M_3^1 is the quotient of $M_{0,c}(S^{p-2}\mathfrak{h})_3 \cong S^3\mathfrak{h}^* \otimes S^{p-2}\mathfrak{h}$ by the image of singular vectors from Lemma 6.4, isomorphic to $S^2\mathfrak{h}^* \otimes S^{p-3}\mathfrak{h}$. The short exact sequence describing this inclusion is the one from Lemma 3.5 combined with Lemma 3.4, giving

$$0 \rightarrow S^2\mathfrak{h}^* \otimes S^{p-3}\mathfrak{h} \rightarrow S^3\mathfrak{h}^* \otimes S^{p-2}\mathfrak{h} \rightarrow S^{p+1}\mathfrak{h} \otimes (\det)^{-3} \rightarrow 0.$$

Under these morphisms, the image of $x_1^2x_2 \otimes y_1^{p-2} \in S^3\mathfrak{h}^* \otimes S^{p-2}\mathfrak{h}$ in $S^{p+1}\mathfrak{h} \otimes (\det)^3$ is $y_1^{p-1}y_2^2$.

Second, M_3^2 is the quotient of $M_3^1 \cong S^{p+1}\mathfrak{h} \otimes (\det)^{-3}$ by the image of singular vectors from Lemma 6.3, which is the space isomorphic to $\mathfrak{h}^* \otimes \mathfrak{h} \otimes (\det)^{-2}$. The short exact sequence realizing this inclusion and quotient is the one from Lemma 3.6, giving

$$0 \rightarrow \mathfrak{h} \otimes \mathfrak{h} \otimes (\det)^{-3} \rightarrow S^{p+1}\mathfrak{h} \otimes (\det)^{-3} \rightarrow S^{p-3}\mathfrak{h} \otimes (\det)^{-1} \rightarrow 0.$$

The image of $y_1^{p-1}y_2^2 \in S^{p+1}\mathfrak{h} \otimes (\det)^{-3}$ under this quotient morphism is $-y_1^{p-3} \in S^{p-3}\mathfrak{h} \otimes (\det)^{-1}$, which is a nonzero element of it.

Hence, the image of $x_1^2x_2 \otimes y_1^{p-2}$ in M_3^2 is the nonzero vector in an irreducible representation. Hence, if we show this vector is singular, it will follow that the entire space M_3^2 consists of singular vectors.

Third, applying Dunkl operators is a map $\mathfrak{h} \otimes M_3^2 \rightarrow M_2^2$, so let us decompose $\mathfrak{h} \otimes M_3^2 \cong \mathfrak{h} \otimes S^{p-3}\mathfrak{h} \otimes (\det)^{-1}$. The short exact sequence doing this is the one from Lemma 3.5,

$$0 \rightarrow S^{p-4}\mathfrak{h} \rightarrow \mathfrak{h} \otimes S^{p-3}\mathfrak{h} \otimes (\det)^{-1} \rightarrow S^{p-2}\mathfrak{h} \otimes (\det)^{-1} \rightarrow 0.$$

As applying Dunkl operator maps this to $M_2^2 \cong S^{p-2}\mathfrak{h} \otimes (\det)^{-1}$, by Schur's lemma the submodule $S^{p-4}\mathfrak{h}$ maps to zero, and the map is zero on $\mathfrak{h} \otimes S^{p-3}\mathfrak{h} \otimes (\det)^{-1}$ if and only if it is zero on the quotient $S^{p-2}\mathfrak{h} \otimes (\det)^{-1}$. Under this quotient map, the vector $y_1 \otimes (-y_1^{p-2})$ maps to $-y_1^{p-2}$, which is a nonzero element of the irreducible representation $S^{p-2}\mathfrak{h} \otimes (\det)^{-1}$. Showing that the entire $\mathfrak{h} \otimes M_3^2$ maps to zero is equivalent to showing that this vector maps to zero, which is equivalent to showing that the image of $D_{y_1}(x_1^2x_2 \otimes y_1^{p-2})$ in M_2^2 is zero.

Finally, we prove $D_{y_1}(x_1^2x_2 \otimes y_1^{p-2})$ is zero in M_2^2 by an explicit calculation using the parametrization of conjugacy classes from Lemma 3.1. The factor of $-c_\lambda$ in $D_{y_1}(x_1^2x_2 \otimes y_1^{p-2})$ is

$$\begin{aligned} & \sum_{b,d} (x_1^2d(\lambda + bd)^2 + x_1x_2(1 - \lambda - bd)(1 + \lambda - bd + 2bd(1 - \lambda - bd)) + \\ & + x_2^2(1 - \lambda - bd)^2b(bd - 1)) \otimes \frac{1}{\lambda^{p-2}} \sum_{i=0}^{p-2} \binom{p-2}{i} (1 - bd)^i d^{p-2-i} y_1^i y_2^{p-2-i}. \end{aligned}$$

The sum is over $b, d \in \mathbb{F}_p$ if $\lambda \neq 1$ and over $b \in \mathbb{F}_p, d \in \mathbb{F}_p^\times$ if $\lambda = 1$. After quotienting out by vectors from the previous two lemmas, whose images in degree 2 are $x_i(x_1 \otimes y_1 + x_2 \otimes y_2)f, f \in S^{p-3}\mathfrak{h}$, and $x_1^2 \otimes y_2^{p-2}, x_2^2 \otimes y_1^{p-2}$, we can write this as

$$\begin{aligned} & \sum_{b,d} x_1x_2 \frac{1}{\lambda^{p-2}} \otimes \left(-d(\lambda + bd)^2 \sum_{i=1}^{p-2} \binom{p-2}{i} (1 - bd)^i d^{p-2-i} y_1^i y_2^{p-2-i} + \right. \\ & + (1 - \lambda - bd)(1 + \lambda - bd + 2bd(1 - \lambda - bd)) \sum_{i=0}^{p-2} \binom{p-2}{i} (1 - bd)^i d^{p-2-i} y_1^i y_2^{p-2-i} - \\ & \left. - (1 - \lambda - bd)^2b(bd - 1) \sum_{i=0}^{p-3} \binom{p-2}{i} (1 - bd)^i d^{p-2-i} y_1^i y_2^{p-2-i} \right) \end{aligned}$$

Reading off the coefficients of $x_1 x_2 \otimes y_1^i y_2^{p-2-i}$ for all $0 \leq i \leq p-2$ and using lemma 3.7 multiple times, we see this is indeed 0. \square

This completes the proof of Proposition 6.1.

6.2. Characters of $L_{t,c}(S^i \mathfrak{h})$ for $i = p-2$ and $t = 1$.

Proposition 6.5. *The reduced character of $L_{1,c}(S^{p-2} \mathfrak{h})$ for generic value of c is*

$$H(z) = [S^{p-2} \mathfrak{h}] + [S^{p-1} \mathfrak{h} \otimes (\det)^{-1}]z + [S^{p-2} \mathfrak{h} \otimes (\det)^{-1}]z^2,$$

so its character is

$$\chi_{L_{1,c}(S^{p-2} \mathfrak{h})}(z) = ([S^{p-2} \mathfrak{h}] + [S^{p-1} \mathfrak{h} \otimes (\det)^{-1}]z^p + [S^{p-2} \mathfrak{h} (\det)^{-1}]z^{2p}) \cdot \chi_{S^{(p)} \mathfrak{h}^*}(z),$$

and its Hilbert series is

$$((p-1) + pz + (p-1)z^{2p}) \cdot \frac{(1-z^p)^2}{(1-z)^2}.$$

Proof. It is explained in Proposition 2.14, Corollary 2.16 and comments between them that the generators of the module $J_{t,c}(\tau)$ for generic c and nonzero t are in degrees divisible by p . If $J_{<mp}$ is the submodule of J generated under $S\mathfrak{h}^*$ by elements of $J_{1,c}(\tau)$ of degrees $< mp$, then the generators in degree mp have nonzero projections to $M_{1,c}(\tau)/J_{<mp}$, they are singular vectors in this quotient, and they form a subrepresentation whose composition factors are a subset of composition factors of $(S^m \mathfrak{h}^*)^p \otimes \tau / ((S^m \mathfrak{h}^*)^p \otimes \tau \cap J_{<mp})$. In Lemmas 6.6, 6.8, and 6.10 below we explicitly find these generators for $\tau = S^{p-2} \mathfrak{h}$, and in Lemmas 6.7 and 6.9 we prove they are the only ones in degrees p , $2p$ and $3p$. The quotient of the Verma module $M_{1,c}(S^{p-2} \mathfrak{h})$ by the submodule generated by these elements is finite dimensional, and zero in degree $4p$, from which we conclude that they generate the whole $J_{1,c}(S^{p-2} \mathfrak{h})$ for generic c , and that this quotient is irreducible.

The reduced character is calculated in the way explained after Proposition 2.14: as we know the generators of $J_{1,c}(S^{p-2} \mathfrak{h})$ explicitly, we evaluate them at $c = 0$ (in fact, they do not depend on c). They are of the form $f_i(x_1^p, x_2^p) \otimes v_i$. The reduced module is then defined to be $S\mathfrak{h}^* \otimes S^{p-2} \mathfrak{h} / \langle f_i(x_1, x_2) \otimes v_i \rangle$. In our case, the generators $f_i(x_1^p, x_2^p) \otimes v_i$ form subrepresentations of type $S^{p-3} \mathfrak{h}$ in degree p , $\mathfrak{h} \otimes (\det)^{-2}$ in degree $2p$, and $S^{p-3} \mathfrak{h} \otimes (\det)^{-1}$ in degree $3p$. The quotient of $S\mathfrak{h}^* \otimes S^{p-2} \mathfrak{h}$ by $\langle f_i(x_1, x_2) \otimes v_i \rangle$ is thus equal to the quotient by subrepresentations of type $S^{p-3} \mathfrak{h}$ in degree 1, $\mathfrak{h} \otimes (\det)^{-2}$ in degree 2, and $S^{p-3} \mathfrak{h} \otimes (\det)^{-1}$ in degree 3, which is easily seen to have the character

$$H(z) = [S^{p-2} \mathfrak{h}] + [S^{p-1} \mathfrak{h} \otimes (\det)^{-1}]z + [S^{p-2} \mathfrak{h} (\det)^{-1}]z^2.$$

\square

Lemma 6.6. *The vectors*

$$x_1^p \otimes y_1 f + x_2^p \otimes y_2 f$$

in $S^p \mathfrak{h}^ \otimes S^{p-2} \mathfrak{h} \cong M_{1,c}(S^{p-2} \mathfrak{h})_p$ are singular in $M_{1,c}(S^{p-2} \mathfrak{h})$ for all $f \in S^{p-3} \mathfrak{h}$. They form a $GL_2(\mathbb{F}_p)$ subrepresentation of $M_{1,c}(S^{p-2} \mathfrak{h})_p$ isomorphic to $S^{p-3} \mathfrak{h}$.*

Proof. The space of these vectors are symmetric with respect to switching indices 1 and 2, so it is enough to prove D_{y_1} acts on it by zero. A computation very similar to the one in Lemma 6.2 gives that the coefficient of $-c_\lambda$ in $D_{y_1}(x_1^p \otimes y_1 f + x_2^p y_2 f)$ is

$$\sum_{b,d} (x_1 + bx_2)^{p-1} \otimes \frac{1}{\lambda} ((1 - \lambda - bd)y_1 + dy_2)(s.f).$$

The sum is over all $b, d \in \mathbb{F}_p$ if $\lambda \neq 1$ and over all $b \in \mathbb{F}_p, d \in \mathbb{F}_p^\times$ if $\lambda = 1$. However, if $\lambda = 1$, then every summand is divisible by d so the term with $d = 0$ does not contribute and we can consider it as a sum over all $d \in \mathbb{F}_p$. The degree in d of every term of this polynomial is less or equal to $1 + \deg f = p - 2 < p - 1$, so by Lemma 3.7, the sum is zero. \square

Lemma 6.7. *For generic c , the vectors from Lemma 6.6 are the only singular vectors in $M_{1,c}(S^{p-2}\mathfrak{h})_p$.*

Proof. We will use Corollary 2.16, which in our situation says that the space of singular vectors in $M_{1,c}(S^{p-2}\mathfrak{h})_p$ has the same composition series (as a $GL_2(\mathbb{F}_p)$ representation) as some subrepresentation of $(\mathfrak{h}^*)^p \otimes S^{p-2}\mathfrak{h}$. This space is isomorphic to $\mathfrak{h}^* \otimes S^{p-2}\mathfrak{h}$ and fits into a short exact sequence from Lemma 3.5

$$0 \rightarrow S^{p-3}\mathfrak{h} \rightarrow \mathfrak{h}^* \otimes S^{p-2}\mathfrak{h} \rightarrow S^{p-1}\mathfrak{h} \otimes (\det)^{-1} \rightarrow 0,$$

so the space of singular vectors in degree p can either be isomorphic to $S^{p-3}\mathfrak{h}$ or to its extension by $S^{p-1}\mathfrak{h} \otimes (\det)^{-1}$.

We are going to show that the quotient of $(\mathfrak{h}^*)^p \otimes S^{p-2}\mathfrak{h}$ by $S^{p-3}\mathfrak{h}$ isomorphic to $S^{p-1}\mathfrak{h} \otimes (\det)^{-1}$ does not consist of singular vectors, and that there is no other composition factor of $S^p\mathfrak{h}^* \otimes S^{p-2}\mathfrak{h}$ isomorphic to $S^{p-1}\mathfrak{h} \otimes (\det)^{-1}$. This will prove the claim.

First, we claim that there is a vector in $(\mathfrak{h}^*)^p \otimes S^{p-2}\mathfrak{h}$ which is not singular. Namely, we claim that $D_{y_1}(x_1^p \otimes y_1^{p-2})$ is not zero. From the calculation in the proof of Lemma 6.6 we can read off that the coefficient of c_1 in it is equal to

$$\sum_{b \in \mathbb{F}_p, d \in \mathbb{F}_p^\times} bd(x_1 + bx_2)^{p-1} \otimes ((1 - bd)y_1 + dy_2)^{p-2},$$

which in turn has a coefficient of $x_1^{p-1} \otimes y_1^{p-2}$ equal to

$$\sum_{b \in \mathbb{F}_p, d \in \mathbb{F}_p^\times} bd(1 - bd)^{p-2} = 1 \neq 0.$$

Second, we claim that $S^p\mathfrak{h}^* \otimes S^{p-2}\mathfrak{h}$ has only one composition factor of type $S^{p-1}\mathfrak{h} \otimes (\det)^{-1}$. The quotient of $S^p\mathfrak{h}^* \otimes S^{p-2}\mathfrak{h}$ by the space $(\mathfrak{h}^*)^p \otimes S^{p-2}\mathfrak{h}$ which we already considered is isomorphic to $S^{p-2}\mathfrak{h} \otimes S^{p-2}\mathfrak{h}$, which by repeated use of Lemma 3.5 has subquotients of the form $S^{2p-4-2j}\mathfrak{h} \otimes (\det)^j$. More precisely, the expression in the Grothendieck group of $GL_2(\mathbb{F}_p)$ representations for $[S^{p-2}\mathfrak{h} \otimes S^{p-2}\mathfrak{h}]$ is

$$[S^{2p-4}\mathfrak{h}] + [S^{2p-6}\mathfrak{h} \otimes (\det)] + \dots + [S^{2p-4-2j}\mathfrak{h} \otimes (\det)^j] + \dots + [S^0\mathfrak{h} \otimes (\det)^{p-2}].$$

Some of these are irreducible (the ones with $2p - 4 - 2j < p$), and the others decompose further using Lemma 3.6 and 3.5

$$\begin{aligned} [S^{2p-4-2j}\mathfrak{h} \otimes (\det)^j] &= [S^{p-4-2j}\mathfrak{h} \otimes \mathfrak{h} \otimes (\det)^j] + [S^{2j+2}\mathfrak{h} \otimes (\det)^{j+1+p-4-2j}] = \\ &= [S^{p-5-2j}\mathfrak{h} \otimes \mathfrak{h} \otimes (\det)^{j+1}] + [S^{p-3-2j}\mathfrak{h} \otimes \mathfrak{h} \otimes (\det)^j] + [S^{2j+2}\mathfrak{h} \otimes (\det)^{j+1+p-4-2j}]. \end{aligned}$$

These are all the composition factors, and none of them is equal to $S^{p-1}\mathfrak{h} \otimes (\det)^{-1}$. \square

Next, we consider the auxiliary module \mathbf{M}^1 , defined as the quotient of the Verma module $M_{1,c}(S^{p-2}\mathfrak{h})$ by the submodule generated by singular vectors $(x_1^p \otimes y_1 + x_2^p \otimes y_2)f$ from Lemma 6.6. This module \mathbf{M}^1 matches $L_{1,c}(S^{p-2}\mathfrak{h})$ in degrees $0, 1, \dots, 2p-1$, and we search for singular vectors in \mathbf{M}_{2p}^1 .

Lemma 6.8. *The images of the vectors*

$$x_1^{2p} \otimes y_2^{p-2}, \quad x_2^{2p} \otimes y_1^{p-2}$$

in \mathbf{M}^1 are singular, and span a subrepresentation isomorphic to $\mathfrak{h} \otimes (\det)^{-2}$.

Proof. The claim that they span a subrepresentation isomorphic to $\mathfrak{h} \otimes (\det)^{-2}$ is easy to check. As this representation is irreducible, it is enough to show that one of them is singular, for example $x_1^{2p} \otimes y_2^{p-2}$. Calculating, as before, the coefficient of $-c_\lambda$ in $D_{y_1}(x_1^{2p} \otimes y_2^{p-2})$, and denoting $\alpha_b = x_1 + bx_2$, we get it is equal to:

$$\begin{aligned} & \sum_{b,d} \frac{1}{\alpha_b} (x_1^{2p} - (x_1 - (1 - \lambda - bd)\alpha_b)^{2p}) \otimes \frac{(b(1 - \lambda - bd)y_1 + (\lambda + bd)y_2)^{p-2}}{\lambda^{p-2}} \\ &= \sum_{b,d} (1 - \lambda - bd) ((1 + \lambda + bd)x_1^p - b(1 - \lambda - bd)x_2^p) \left(\sum_{i=0}^{p-1} (-1)^i b^i x_1^{p-1-i} x_2^i \right) \otimes \\ & \quad \otimes \frac{1}{\lambda^{p-2}} \sum_{j=0}^{p-2} \binom{p-2}{j} b^j (1 - \lambda - bd)^j (\lambda + bd)^{p-2-j} y_1^j y_2^{p-2-j}. \end{aligned}$$

Summing over all d and using lemma 3.7, this is equal to

$$\begin{aligned} & \sum_b (2\lambda b x_1^p - 2b^2(1 - \lambda)x_2^p) \left(\sum_{i=0}^{p-1} (-1)^i b^i x_1^{p-1-i} x_2^i \right) \otimes \frac{1}{\lambda^{p-2}} \sum_{j=0}^{p-2} \binom{p-2}{j} (-1)^j b^{j+p-2} y_1^j y_2^{p-2-j} + \\ & + \sum_b (b^2 x_1^p + b^3 x_2^p) \left(\sum_{i=0}^{p-1} (-1)^i b^i x_1^{p-1-i} x_2^i \right) \otimes \frac{1}{\lambda^{p-2}} \left(-2\lambda b^{p-3} y_2^{p-2} - 2(1 - \lambda) b^{2p-5} y_1^{p-2} + \right. \\ & \quad \left. + \sum_{j=1}^{p-3} \binom{p-2}{j} (-1)^j b^{p-3-j} (-2\lambda - j) y_1^j y_2^{p-2-j} \right). \end{aligned}$$

Summing over $b \in \mathbb{F}_p$ using lemma 3.7 and reorganizing the terms, we get

$$\begin{aligned} & \frac{-1}{\lambda^{p-2}} \left(2(1 - \lambda) (x_1^{2p-2} x_2 - x_1^{p-1} x_2^p) \otimes y_1^{p-2} + 2\lambda (x_1 x_2^{2p-2} - x_1^p x_2^{p-1}) \otimes y_2^{p-2} + \right. \\ & + 2\lambda x_1^{2p-2} x_2 \otimes y_1^{p-2} + 2\lambda x_1^p x_2^{p-1} \otimes y_2^{p-2} + 2(1 - \lambda) x_1^{p-1} x_2^p \otimes y_1^{p-2} + 2(1 - \lambda) x_1 x_2^{2p-2} \otimes y_2^{p-2} + \\ & \quad \left. + \sum_{i=1}^{p-3} \binom{p-2}{i} ((2+i)x_1^{2p-2-i} x_2^{i+1} - i x_1^{p-1-i} x_2^{p+i}) \otimes y_1^{p-2-i} y_2^i \right). \end{aligned}$$

This is nonzero in $M_{1,c}(S^{p-2}\mathfrak{h})$, and the vectors from lemma are not singular there. However, in the quotient \mathbf{M}^1 , this expression is zero. This can be shown by replacing, in the above long expression, anything of the form $a \cdot x_2^p \otimes y_2 \cdot f$ (meaning any term whose degree of x_2 is at least p and whose degree of y_2 is at least 1) by the equivalent expression $-a \cdot x_1^p \otimes y_1 \cdot f$. The expression then simplifies to 0, showing that in the quotient \mathbf{M}^1 , $D_{y_1}(x_1^{2p} \otimes y_2^{p-2}) = 0$.

Similarly, the coefficient in $D_{y_2}(x_1^{2p} \otimes y_2^{p-2})$ of $-c_\lambda$ is equal to

$$\sum_{b,d} \frac{b}{\alpha_b} (x_1^{2p} - (x_1 - (1 - \lambda - bd)\alpha_b)^{2p}) \otimes \frac{(b(1 - \lambda - bd)y_1 + (\lambda + bd)y_2)^{p-2}}{\lambda^{p-2}} +$$

$$+ \sum_a \frac{1}{x_2} (x_1^{2p} - (x_1 - ax_2)^{2p}) \otimes \frac{(ay_1 + y_2)^{p-2}}{\lambda^{p-2}}.$$

The proof that the first part of the sum is 0 in \mathbf{M}^1 is very similar to the previous computation, as it differs from the expression calculated there just by one power of b . The second part is equal to

$$\frac{-1}{\lambda^{p-2}} (2x_1^p x_2^{p-1} \otimes y_1^{p-2} + 2x_2^{2p-1} \otimes y_1^{p-3} y_2),$$

which is also zero in \mathbf{M}^1 . □

Lemma 6.9. *For generic c , the vectors from Lemma 6.8 are the only singular vectors in \mathbf{M}_{2p}^1 .*

Proof. The space of p -th powers in $M_{1,c}(S^{p-2}\mathfrak{h})_{2p}$ is isomorphic to $S^2\mathfrak{h}^* \otimes S^{p-2}\mathfrak{h} \subseteq S^{2p}\mathfrak{h}^* \otimes S^{p-2}\mathfrak{h}$. Using Lemma 3.5, its image in the quotient \mathbf{M}_{2p}^1 is isomorphic to the quotient of $S^2\mathfrak{h} \otimes S^{p-2}\mathfrak{h} \otimes (\det)^{-2}$ by $\mathfrak{h} \otimes S^{p-3}\mathfrak{h} \otimes (\det)^{-1}$, which is $S^p\mathfrak{h} \otimes (\det)^{-2}$. So this is the maximal possible space of singular vectors in \mathbf{M}_{2p}^1 , and the space of singular vectors in \mathbf{M}_{2p}^1 is its subspace.

This space decomposes by Lemma 3.6 as

$$0 \rightarrow \mathfrak{h} \otimes (\det)^{-2} \rightarrow S^p\mathfrak{h} \otimes (\det)^{-2} \rightarrow S^{p-2}\mathfrak{h} \otimes (\det)^{-1} \rightarrow 0.$$

We already showed in that the subspace $\mathfrak{h} \otimes (\det)^{-2}$ consists of singular vectors; to prove that not the entire $S^p\mathfrak{h} \otimes (\det)^{-2}$ does, it is enough to find one nonsingular vector. By another explicit computation, one can show that $x_1^p x_2^p \otimes y_1^{p-1}$ is a p -th power that is not annihilated by D_{y_1} .

Finally, we need to show that there is no other composition factor of \mathbf{M}_{2p}^1 made of singular vectors. If such a composition factor existed, it would have to be isomorphic to $S^{p-2}\mathfrak{h} \otimes (\det)^{-1}$, and show up as a submodule of the quotient of \mathbf{M}_{2p}^1 by the space of p -th powers which we already considered.

Using Lemmas 3.5 and 3.6, this space is isomorphic to

$$S^{p-2}\mathfrak{h} \otimes S^{p-1}\mathfrak{h} \otimes (\det)^{-1}.$$

Using [5], Theorem (5.3), this is isomorphic to

$$S^{p(p-1)-1}\mathfrak{h} \otimes (\det)^{-1}.$$

The only possible space of singular vectors would be isomorphic to $S^{p-2}\mathfrak{h} \otimes (\det)^{-1}$ and contained in the socle (maximal semisimple submodule) of $S^{p(p-1)-1}\mathfrak{h} \otimes (\det)^{-1}$. However, Theorem (5.9) in [5] show that this socle is in fact

$$\bigoplus_{m=0}^{(p-3)/2} S^{2m+1}\mathfrak{h} \otimes (\det)^{p-3-m}.$$

As these are all irreducible modules and none of them is isomorphic to $S^{p-2}\mathfrak{h} \otimes (\det)^{-1}$, we conclude there are no other singular vectors in \mathbf{M}_{2p}^1 . □

The second auxiliary module \mathbf{M}^2 to consider is the quotient of \mathbf{M}^1 by the rational Cherednik algebra submodule generated by the two dimensional space of singular vectors from the previous lemma. Because of Corollary 2.16 and the previous lemmas in this section (Lemma

6.7 and 6.9), \mathbf{M}^2 is equal to $L_{1,c}(S^{p-2}\mathfrak{h})$ in degrees up to $3p-1$. In degree $3p$, \mathbf{M}^2 contains some new singular vectors, given in the following lemma.

Lemma 6.10. *All the vectors of the form $(S^3\mathfrak{h}^*)^p \otimes S^{p-2}\mathfrak{h} \subseteq M_{1,c}(S^{p-2}\mathfrak{h})_{3p}$ are in $\text{Ker } B$.*

Proof. The space of p -th powers in $M_{1,c}(S^{p-2}\mathfrak{h})_{3p}$ is $(S^3\mathfrak{h}^*)^p \otimes S^{p-2}\mathfrak{h}$. The quotient of this space by the subspace $(S^2\mathfrak{h}^*)^p \otimes S^{p-3}\mathfrak{h}$ generated by singular vectors from Lemma 6.6 is by Lemma 3.5 isomorphic to $S^{p+1}\mathfrak{h} \otimes (\det)^{-3}$. The quotient of this space by the subspace $(\mathfrak{h}^*)^p \otimes \mathfrak{h} \otimes (\det)^{-2}$ generated by singular vectors from Lemma 6.8 is by Lemma 3.6 isomorphic to $S^{p-3}\mathfrak{h} \otimes (\det)^{-1}$, and this is the space of p -th powers in the \mathbf{M}_{3p}^2 .

We are going to show that this space is made of singular vectors by showing that the restriction to $\mathfrak{h} \otimes S^{p-3}\mathfrak{h} \otimes (\det)^{-1}$ of the map

$$\mathfrak{h} \otimes \mathbf{M}_{3p}^2 \rightarrow \mathbf{M}_{3p-1}^2$$

given by $y \otimes m \mapsto D_y(m)$, which is a homomorphism of group representations, has to be zero.

The source space of this map is $\mathfrak{h} \otimes S^{p-3}\mathfrak{h} \otimes (\det)^{-1}$, which fits into the short exact sequence

$$0 \rightarrow S^{p-4}\mathfrak{h} \rightarrow \mathfrak{h} \otimes S^{p-3}\mathfrak{h} \otimes (\det)^{-1} \rightarrow S^{p-2}\mathfrak{h} \otimes (\det)^{-1} \rightarrow 0.$$

The image of the map is a subrepresentation of the target space \mathbf{M}_{3p-1}^2 , so if we show that the socle of \mathbf{M}_{3p-1}^2 does not have $S^{p-4}\mathfrak{h}$ nor $S^{p-2}\mathfrak{h} \otimes (\det)^{-1}$ as direct summands, it will follow that the map is zero.

By applying Lemma 3.6 twice and Lemma 3.5 once, we see that the quotient of \mathbf{M}_{3p-1}^2 by the image $S^{2p-1}\mathfrak{h}^* \otimes S^{p-3}\mathfrak{h}$ of singular vectors from Lemma 6.6 is isomorphic to $S^{p-1}\mathfrak{h} \otimes S^p\mathfrak{h} \otimes (\det)^{-2}$. The quotient of that by the image $S^{p-1}\mathfrak{h}^* \otimes \mathfrak{h} \otimes (\det)^{-2}$ is by Lemma 3.6 isomorphic to $S^{p-1}\mathfrak{h} \otimes S^{p-2}\mathfrak{h} \otimes (\det)^{-1}$. Using [5], Theorem (5.3) again, this is isomorphic to $S^{p(p-1)-1}\mathfrak{h} \otimes (\det)^{-1}$, whose socle is by Theorem (5.9) in [5] again equal to

$$\bigoplus_{m=0}^{(p-3)/2} S^{2m+1}\mathfrak{h} \otimes (\det)^{p-3-m}.$$

None of these summands is of the type $S^{p-4}\mathfrak{h}$ nor $S^{p-2}\mathfrak{h} \otimes (\det)^{-1}$, so the required map is zero. This proves the lemma. \square

7. CHARACTERS OF $L_{t,c}(S^i\mathfrak{h})$ FOR $i = p-1$

7.1. Characters of $L_{t,c}(S^i\mathfrak{h})$ for $i = p-1$ and $t = 0$. In this section we will calculate the character of $L_{0,c}(S^{p-1}\mathfrak{h})$ for generic c . We first find a certain space $\text{span}_{\mathbb{k}}\{v_0, \dots, v_{p-1}\}$ of singular vectors in $M_{0,c}(S^{p-1}\mathfrak{h})_{p-1}$. We define an auxiliary module M as a quotient of $M_{0,c}(S^{p-1}\mathfrak{h})$ by the $H_{0,c}(GL_2, \mathfrak{h})$ -submodule generated by these singular vectors and by the action of the algebra of invariants $(S\mathfrak{h}^*)_+^G = \mathbb{k}[Q_0, Q_1]_+$ (for definitions of Q_0 and Q_1 , see section 3.1). We calculate the character of M , and finally we show that M is irreducible and isomorphic to $L_{0,c}(S^{p-1}\mathfrak{h})$.

We will extensively use Corollary 4.6, which states that all the composition factors of $M_{t,c}(S^{p-1}\mathfrak{h})$ are of the form $L_{t,c}(S^{p-1}\mathfrak{h})$. Because of that, all the singular vectors in $M_{t,c}(S^{p-1}\mathfrak{h})$ or any of its subrepresentations or quotients have all of their composition factors isomorphic to $S^{p-1}\mathfrak{h}$.

First, we find singular vectors in degree $p-1$.

Lemma 7.1. *The vector*

$$x_1^{p-1} \otimes y_2^{p-1} - x_2^{p-1} \otimes y_1^{p-1}$$

in $S^{p-1}\mathfrak{h}^ \otimes S^{p-1}\mathfrak{h} \cong M_{0,c}(S^{p-1}\mathfrak{h})_{p-1}$ is singular.*

Proof. The proof is pure computation, using the parametrization of conjugacy classes from Lemma 3.1 and Lemma 3.7 extensively.

First, this vector is antisymmetric with respect to indices 1, 2, so it is enough to show that D_{y_1} acts on it by 0. For any λ the coefficient of $-c_\lambda$ in $D_{y_1}(x_1^{p-1} \otimes y_2^{p-1} - x_2^{p-1} \otimes y_1^{p-1})$ is

$$(\star) \quad \sum_{s \in C_\lambda} (y_1, \alpha_s) \left(\frac{x_1^{p-1} - s.x_1^{p-1}}{\alpha_s} \otimes (s.y_2)^{p-1} - \frac{x_2^{p-1} - s.x_2^{p-1}}{\alpha_s} \otimes (s.y_1)^{p-1} \right).$$

Let us rewrite this using the parametrization of C_λ from Lemma 3.1. We use notation $\alpha_b = \begin{bmatrix} 1 \\ b \end{bmatrix} \in \mathfrak{h}^*$. The sum is over all $b, d \in \mathbb{F}_p$ if $\lambda \neq 1$ and over all $b, d \in \mathbb{F}_p$, $d \neq 0$, if $\lambda = 1$. The above expression is equal to:

$$\begin{aligned} (\star) &= \sum_{b,d} \frac{1}{\alpha_b} (x_1^{p-1} - (x_1 - (1 - \lambda - bd)\alpha_b)^{p-1}) \otimes \frac{1}{\lambda^{p-1}} (b(1 - \lambda - bd)y_1 + (\lambda + bd)y_2)^{p-1} + \\ &\quad + \frac{1}{\alpha_b} ((x_2 - d\alpha_b)^{p-1} - x_2^{p-1}) \otimes \frac{1}{\lambda^{p-1}} ((1 - bd)y_1 + dy_2)^{p-1} = \\ &= \frac{1}{\lambda^{p-1}} \sum_{b,d} \sum_{i=1}^{p-1} \binom{p-1}{i} (-1)^{i+1} x_1^{p-1-i} (1 - \lambda - bd)^i \alpha_b^{i-1} \otimes (b(1 - \lambda - bd)y_1 + (\lambda + bd)y_2)^{p-1} + \\ &\quad + \binom{p-1}{i} (-1)^i x_2^{p-1-i} d^i \alpha_b^{i-1} \otimes ((1 - bd)y_1 + dy_2)^{p-1} = \\ &= \frac{1}{\lambda^{p-1}} \sum_{b,d} \sum_{i=1}^{p-1} \sum_{j=0}^{i-1} \sum_{k=0}^{p-1} \binom{p-1}{i} \binom{i-1}{j} \binom{p-1}{k} \cdot \\ &\quad \cdot \left((-1)^{i+1} (1 - \lambda - bd)^i b^j x_1^{p-j-2} x_2^j \otimes b^{p-1-k} (1 - \lambda - bd)^{p-1-k} (\lambda + bd)^k y_1^{p-1-k} y_2^k + \right. \\ &\quad \left. + (-1)^i d^i b^{i-1-j} x_1^j x_2^{p-j-2} \otimes (1 - bd)^{p-1-k} d^k y_1^{p-1-k} y_2^k \right) = \\ &= \frac{1}{\lambda^{p-1}} \sum_{k=0}^{p-1} \sum_{j=0}^{p-2} \binom{p-1}{k} x_1^{p-j-2} x_2^j \otimes y_1^{p-1-k} y_2^k \cdot \\ &\quad \cdot \sum_{b,d} \left(\sum_{i=j+1}^{p-1} \binom{i-1}{j} \binom{p-1}{i} (-1)^{i+1} (1 - \lambda - bd)^{p-1-k+i} (\lambda + bd)^k b^{p-1-k+j} + \right. \\ &\quad \left. + \sum_{i=p-1-j}^{p-1} \binom{p-1}{i} \binom{i-1}{p-2-j} (-1)^i d^{i+k} b^{i+1+j-p} (1 - bd)^{p-1-k} \right). \end{aligned}$$

Reading off the coefficient of $x_1^{p-j-2} x_2^j \otimes y_1^{p-1-k} y_2^k$ and using that $\frac{1}{\lambda^{p-1}} \binom{p-1}{k}$ is never zero, the claim that $(\star) = 0$ is equivalent to showing that for every $0 \leq k \leq p-1$, $0 \leq j \leq p-2$,

the expression $(\star\star)$ is zero, where $(\star\star)$ is

$$\begin{aligned}
& \sum_{b,d} \left(\sum_{i=j+1}^{p-1} \binom{i-1}{j} \binom{p-1}{i} (-1)^{i+1} (1-\lambda-bd)^{p-1-k+i} (\lambda+bd)^k b^{p-1-k+j} + \right. \\
& \quad \left. + \sum_{i=p-1-j}^{p-1} \binom{p-1}{i} \binom{i-1}{p-2-j} (-1)^i d^{i+k} b^{i+1+j-p} (1-bd)^{p-1-k} \right) \\
&= \sum_{b,d} \left(\sum_{i=j+1}^{p-1} \binom{i-1}{j} \binom{p-1}{i} (-1)^{i+1} (1-\lambda-bd)^{p-1-k+i} (\lambda+bd)^k b^{p-1-k+j} + \right. \\
& \quad \left. + \sum_{i=0}^j \binom{p-1}{p-1-i} \binom{p-2-i}{p-2-j} (-1)^i d^{p-1+k-i} b^{j-i} (1-bd)^{p-1-k} \right) \\
&= \sum_{b,d} \left(\sum_{i=j+1}^{p-1} \sum_{m=0}^{p-1-k+i} \sum_{n=0}^k \binom{i-1}{j} \binom{p-1}{i} \binom{p-1-k+i}{m} \binom{k}{n} (-1)^{m+i+1} \cdot \right. \\
& \quad \cdot (1-\lambda)^{p-1-k+i-m} \lambda^{k-n} b^{m+n+p-1-k+j} d^{m+n} + \\
& \quad \left. \sum_{i=0}^j \sum_{l=0}^{p-1-k} \binom{p-1}{p-1-i} \binom{p-2-i}{p-2-j} \binom{p-1-k}{l} (-1)^{i+l} b^{j-i+l} d^{p-1+k-i+l} \right)
\end{aligned}$$

Now we will use Lemma 3.7, which states that $\sum_{b \in \mathbb{F}_p} b^N = 0$ and $\sum_{d \in \mathbb{F}_p} b^N = 0$, unless $N \equiv 0 \pmod{p-1}$.

First assume $\lambda \neq 1$. The first part of the sum includes $\sum_b b^{m+n+p-1-k+j}$ and $\sum_d d^{m+n}$, so it is zero unless

$$m+n \equiv 0 \pmod{p-1}$$

$$m+n+p-1-k+j \equiv 0 \pmod{p-1},$$

which implies

$$j \equiv k \pmod{p-1}.$$

The second part of the sum includes $\sum_{b,d} b^{j-i+l} d^{p-1+k-i+l}$, so it is zero unless

$$j-i+l \equiv 0 \pmod{p-1}$$

$$p-1+k-i+l \equiv 0 \pmod{p-1},$$

which again implies

$$j \equiv k \pmod{p-1}.$$

As $0 \leq k \leq p-1$, $0 \leq j \leq p-2$, the possibilities for $j \equiv k \pmod{p-1}$ are $j=0, k=p-1$ or $j=k$. Let us calculate $(\star\star)$ in those two cases separately.

If $j = 0, k = p - 1$, then

$$\begin{aligned}
(\star\star) &= \sum_{b,d} \sum_{i=1}^{p-1} \sum_{m=0}^i \sum_{n=0}^{p-1} \binom{p-1}{i} \binom{i}{m} \binom{p-1}{n} (-1)^{m+i+1} (1-\lambda)^{i-m} \lambda^{p-1-n} b^{m+n} d^{m+n} + \\
&\quad + \sum_{b,d} d^{2(p-1)} = \\
&= \sum_{b,d} \sum_{i=1}^{p-1} \sum_{m=0}^i \sum_{n=0}^{p-1} \binom{p-1}{i} \binom{i}{m} \binom{p-1}{n} (-1)^{m+i+1} (1-\lambda)^{i-m} \lambda^{p-1-n} b^{m+n} d^{m+n} = \\
&= \sum_{b,d} (\lambda - bd)^{2(p-1)} - (\lambda - bd)^{p-1} = 0.
\end{aligned}$$

If $j = k$, then, using that $a^p = a$,

$$\begin{aligned}
(\star\star) &= \sum_{b,d} \left(\sum_{i=j+1}^{p-1} \binom{i-1}{j} \binom{p-1}{i} (-1)^{i+1} (1-\lambda-bd)^{i-j} b^{p-1} (\lambda+bd)^j + \right. \\
&\quad \left. + \sum_{i=0}^j \binom{p-1}{i} \binom{p-2-i}{p-2-j} (-1)^i d^{p-1+j-i} b^{j-i} (1-bd)^{p-1-j} \right) \\
&= \sum_{b,d} \sum_{i=j+1}^{p-1} \sum_{m=0}^{i-j} \sum_{n=0}^j \binom{i-1}{j} \binom{p-1}{i} \binom{i-j}{m} \binom{j}{n} (1-\lambda)^{i-j-m} \lambda^{j-n} (-1)^{m+i+1} b^{p-1+m+n} d^{m+n} + \\
&\quad + \sum_{b,d} \sum_{i=0}^j \sum_{l=0}^{p-1-j} \binom{p-1}{i} \binom{p-2-i}{p-2-j} \binom{p-1-j}{l} (-1)^{l+i} b^{j-i+l} d^{p-1+j-i+l}
\end{aligned}$$

Again using that $\sum_{b \in \mathbb{F}_p} b^N = 0$ unless $N \equiv 0 \pmod{p-1}$, $N \neq 0$, this is equal to:

$$\begin{aligned}
(\star\star) &= \sum_{b,d} \binom{p-2}{j} (-1)^{j+1} b^{2(p-1)} d^{p-1} + \sum_{b,d} \binom{p-2}{j} (-1)^j b^{p-1} d^{2(p-1)} \\
&= 0.
\end{aligned}$$

Let us now do a very similar computation for $\lambda = 1$. Now $\sum_{b,d}$ is over $b, d \in \mathbb{F}_p$, $d \neq 0$.

$$\begin{aligned}
(\star\star) &= \sum_{b,d} \left(\sum_{i=j+1}^{p-1} \sum_{n=0}^k \binom{i-1}{j} \binom{p-1}{i} \binom{k}{n} (-1)^{p-k} b^{2(p-1-k)+i+n+j} d^{p-1-k+i+n} + \right. \\
&\quad \left. + \sum_{i=0}^j \sum_{l=0}^{p-1-k} \binom{p-1}{p-1-i} \binom{p-2-i}{p-2-j} \binom{p-1-k}{l} (-1)^{i+l} b^{j-i+l} d^{p-1+k-i+l} \right)
\end{aligned}$$

Again, this is zero unless $j \equiv k \pmod{p-1}$. If $j = 0, k = p-1$, it is equal to

$$\begin{aligned}
(\star\star) &= \sum_{b,d} \sum_{i=1}^{p-1} \sum_{n=0}^{p-1} \binom{p-1}{i} \binom{p-1}{n} (-1)^{i+n} d^{i+n} + \sum_{b,d} b^0 d^{2(p-1)} \\
&= - \sum_{b,d} \left(\sum_{i=1}^{p-1} \binom{p-1}{i} \binom{p-1}{p-1-i} b^{p-1} d^{p-1} + b^{2(p-1)} d^{2(p-1)} \right) \\
&= - \sum_{i=0}^{p-1} \binom{p-1}{i}^2 \sum_{b,d} b^{p-1} d^{p-1} \\
&= - \sum_{i=0}^{p-1} \binom{p-1}{i}^2 = 0.
\end{aligned}$$

If $j = k$,

$$\begin{aligned}
(\star\star) &= \sum_{b,d} \left(\sum_{i=j+1}^{p-1} \sum_{n=0}^j \binom{i-1}{j} \binom{p-1}{i} \binom{j}{n} (-1)^{p-j} b^{2(p-1)-j+i+n} d^{p-1-j+i+n} + \right. \\
&\quad \left. + \sum_{i=0}^j \sum_{l=0}^{p-1-j} \binom{p-1}{p-1-i} \binom{p-2-i}{p-2-j} \binom{p-1-j}{l} (-1)^{i+l} b^{j-i+l} d^{p-1+j-i+l} \right) \\
&= \sum_{b,d} \binom{p-2}{j} (-1)^{p-j} b^{3(p-1)} d^{2(p-1)} + \\
&\quad + \sum_{b,d} \binom{p-2}{p-2-j} (-1)^{p-1-j} b^{p-1} d^{2(p-1)} = \\
&= \binom{p-2}{j} (-1)^{p-j} + \binom{p-2}{p-2-j} (-1)^{p-1-j} = 0.
\end{aligned}$$

So, $(\star\star) = 0$ and the vector $x_1^{p-1} \otimes y_2^{p-1} - x_2^{p-1} \otimes y_1^{p-1}$ is singular. □

Lemma 7.2. *There are no singular vectors in $M_{0,c}(S^{p-1}\mathfrak{h})_i$ for $i < p-1$, and the space of singular vectors in $M_{0,c}(S^{p-1}\mathfrak{h})_{p-1}$ is isomorphic to $S^{p-1}\mathfrak{h}$ as a $GL_2(\mathbb{F}_p)$ -representation.*

Proof. From the previous lemma it follows that the space of singular vectors in $M_{0,c}(S^{p-1}\mathfrak{h})_{p-1}$ is nonzero, and from Lemma 4.6 that all the composition factors of it are isomorphic to $S^{p-1}\mathfrak{h}$. We will now show that for $0 \leq i \leq p-1$, $M_{0,c}(S^{p-1}\mathfrak{h})_i \cong S^i\mathfrak{h}^* \otimes S^{p-1}\mathfrak{h}$ has no composition factors isomorphic to $S^{p-1}\mathfrak{h}$ unless $i = 0$ or $i = p-1$, in which case it has one. The claim follows from this.

Using Proposition 3.5 about the tensor products of symmetric powers, and calculating in the Grothendieck group of the category of finite dimensional representations of the group $GL_2(\mathbb{F}_p)$ (so, disregarding the question whether the short exact sequence is split or not), we get:

$$\begin{aligned}
[S^i\mathfrak{h}^* \otimes S^{p-1}\mathfrak{h}] &= [S^i\mathfrak{h} \otimes S^{p-1}\mathfrak{h} \otimes (\det)^{-i}] \\
&= [S^{p+i-1}\mathfrak{h} \otimes (\det)^{-i}] + [S^{p+i-3}\mathfrak{h} \otimes (\det)^{-i+1}] + \dots + [S^{p-1-i}\mathfrak{h}].
\end{aligned}$$

The representation $S^{p-1}\mathfrak{h}$ only appears on this list of representations $S^{p+i-1-2j}\mathfrak{h} \otimes (\det)^{-i+j}$ for $0 \leq j \leq i$, when $i = 0$, which is the trivial case. However, some of the representations on the list are reducible, namely the ones with $i-1-2j \geq 0$. Decomposing them by Proposition 3.6,

$$\begin{aligned} [S^{p+i-1-2j}\mathfrak{h} \otimes (\det)^{-i+j}] &= \\ &= [S^{i-1-2j}\mathfrak{h} \otimes \mathfrak{h} \otimes (\det)^{-i+j}] + [S^{p-1-i+2j}\mathfrak{h} \otimes (\det)^{-j}] \\ &= [S^{i-2-2j}\mathfrak{h} \otimes (\det)^{-i+j+1}] + [S^{i-2j}\mathfrak{h} \otimes (\det)^{-i+j}] + [S^{p-1-i+2j}\mathfrak{h} \otimes (\det)^{-j}] \end{aligned}$$

Here, we follow the convention that $S^k\mathfrak{h} = 0$ if $k < 0$. In this decomposition all representations are irreducible. Using that $i-1-2j \geq 0$, $p-1 \geq i \geq 0$ and $i \geq j \geq 0$, we see that $S^{p-1}\mathfrak{h}$ appears on this list only once, namely when $j = 0$, $i = p-1$. \square

This space is generated by the singular vector from Lemma 7.1. Its explicit basis, which we will need in computations below, is given by $v_0, \dots, v_{p-1} \in S^{p-1}\mathfrak{h}^* \otimes S^{p-1}\mathfrak{h}$, where

$$v_k = \sum_{i=0}^k (-1)^i x_1^{k-i} x_2^{p-1-k+i} \otimes y_1^{p-1-i} y_2^i + \sum_{i=k}^{p-1} (-1)^i x_1^{p-1+k-i} x_2^{i-k} \otimes y_1^{p-1-i} y_2^i.$$

Remember from section 3.1 that the algebra of invariants $(S\mathfrak{h}^*)^{GL_2(\mathbb{F}_p)}$ is a polynomial algebra generated by polynomials Q_0 and Q_1 of degrees p^2-p and p^2-1 . They are constructed explicitly as:

$$\begin{aligned} Q_0 &= \begin{vmatrix} x_1^p & x_2^p \\ x_1 & x_2 \end{vmatrix}^{p-1} = (x_1^p x_2 - x_1 x_2^p)^{p-1} = (x_1 x_2 (x_1^{p-1} - x_2^{p-1}))^{p-1} \\ Q_1 &= \frac{\begin{vmatrix} x_1^{p^2} & x_2^{p^2} \\ x_1 & x_2 \end{vmatrix}}{\begin{vmatrix} x_1^p & x_2^p \\ x_1 & x_2 \end{vmatrix}} = \frac{x_1^{p^2} x_2 - x_1 x_2^{p^2}}{x_1^p x_2 - x_1 x_2^p} = \frac{x_1^{p^2-1} - x_2^{p^2-1}}{x_1^{p-1} - x_2^{p-1}} = \sum_{i=0}^p x_1^{(p-1)i} x_2^{(p-1)(p-i)}. \end{aligned}$$

Alternatively, the determinant $\begin{vmatrix} x_1^p & x_2^p \\ x_1 & x_2 \end{vmatrix}$ can be described as a product of all the linear polynomials of the form $x_1 + ax_2$ and x_2 . Similarly, Q_0 is the product of all the nonzero linear polynomials $ax_1 + bx_2$, and Q_1 is the product of all the irreducible monic quadratic polynomials $x_1^2 + ax_1 x_2 + bx_2^2$.

Also remember that at $t = 0$, the space $(S\mathfrak{h}^*)_+^{GL_2(\mathbb{F}_p)} \otimes \tau \subseteq M_{0,c}(\tau)$ is always a subspace of $J_{0,c}(\tau)$, and that the spaces $Q_1 \otimes \tau$ and $Q_0 \otimes \tau$ consist of singular vectors.

Let us consider three spaces of singular vectors in $M_{0,c}(S^{p-1}\mathfrak{h})$, all isomorphic to $S^{p-1}\mathfrak{h}$: $\text{span}\{v_0, \dots, v_{p-1}\}$ in degree $p-1$, $Q_1 \otimes S^{p-1}\mathfrak{h}$ in degree p^2-p , and $Q_0 \otimes S^{p-1}\mathfrak{h}$ in degree p^2-1 . We want to study the $H_{0,c}(GL_2, \mathfrak{h})$ -submodule of $M_{0,c}(S^{p-1}\mathfrak{h})$ generated by these vectors, and calculate the character of the quotient M of $M_{0,c}(S^{p-1}\mathfrak{h})$ by this submodule.

Proposition 7.3. *Let V be the $H_{0,c}(GL_2(\mathbb{F}_p))$ submodule of $M_{0,c}(S^{p-1}\mathfrak{h})$ generated by singular vectors v_0, \dots, v_{p-1} . Then*

$$Q_1 \otimes S^{p-1}\mathfrak{h} \subseteq V,$$

while the intersection of the submodule generated by $Q_0 \otimes S^{p-1}\mathfrak{h}$ and V is generated by $Q_0 v_0, \dots, Q_0 v_{p-1}$ in degree $(p^2-1)(p-1)$.

Proof. Let $l = 0$ or 1 , and let us study the intersection of the submodule generated by $Q_l \otimes S^{p-1}\mathfrak{h}$ and V . This is a graded submodule of $M_{0,c}(S^{p-1}\mathfrak{h})$, with elements of the form

$$h_0 v_0 + h_1 v_1 + \dots h_{p-1} v_{p-1} = Q_l f,$$

where $h_i(x_1, x_2) \in S^n \mathfrak{h}^*$ for some $n \geq 0$, and $f \in S^{n+p-1-\deg(Q_l)} \mathfrak{h}^* \otimes S^{p-1} \mathfrak{h}$.

This is a linear equation in $S\mathfrak{h}^* \otimes S^{p-1}\mathfrak{h}$ with unknowns h_i and f . Reading off the coefficients with $y_1^{p-1-i} y_2^i \in S^{p-1}\mathfrak{h}$ in this equation, we can think of it as a system of p linear equations in $S\mathfrak{h}$, with unknowns h_i and $f_i \in S\mathfrak{h}^*$, $f = \sum_i f_i \otimes y_1^{p-1-i} y_2^i$. The left hand side can then be written as

$$\begin{bmatrix} x_1^{p-1} + x_2^{p-1} & x_1 x_2^{p-2} & \dots & x_1^k x_2^{p-1-k} & \dots & x_1^{p-1} \\ -x_1^{p-2} x_2 & -(x_1^{p-1} + x_2^{p-1}) & \dots & -x_1^{k-1} x_2^{p-k} & \dots & -x_1^{p-2} x_2 \\ x_1^{p-3} x_2^2 & x_1^{p-2} x_2 & \dots & x_1^{k-2} x_2^{p-k+1} & \dots & x_1^{p-3} x_2^2 \\ \vdots & \vdots & & \vdots & & \vdots \\ (-1)^k x_1^{p-1-k} x_2^k & (-1)^k x_1^{p-k} x_2^{k-1} & \dots & (-1)^k (x_1^{p-1} + x_2^{p-1}) & \dots & x_1^{p-1-k} x_2^k \\ \vdots & \vdots & & \vdots & & \vdots \\ x_2^{p-1} & x_1 x_2^{p-2} & \dots & x_1^k x_2^{p-1-k} & \dots & x_1^{p-1} + x_2^{p-1} \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_k \\ \vdots \\ h_{p-1} \end{bmatrix}.$$

The i -th row represents the coefficient of $y_1^{p-1-i} y_2^i$, and the k -th column corresponds to v_k . Call the matrix of this system \mathbf{A} , denote the vector with entries h_i by \vec{h} and the vector with entries f_i by \vec{f} . The system of equations in matrix form can then be written as

$$\mathbf{A} \vec{h} = Q_l \vec{f}.$$

Next, we need a lemma.

Lemma 7.4. $\det \mathbf{A} = (-1)^{p-1/2} Q_1$.

Proof. Factoring out the coefficient -1 from all even rows accounts for the sign $(-1)^{(p-1)/2}$. The remaining matrix \mathbf{A}' has $x_1^{p-1} + x_2^{p-1}$ on the main diagonal, $x_1 x_2^{p-2}$ above the main diagonal, $x_1^2 x_2^{p-3}$ above that etc, and $x_1^{p-2} x_2$ below the main diagonal, $x_1^{p-3} x_2^2$ below that, etc.

To show $\det \mathbf{A}'$ is invariant under $GL_2(\mathbb{F}_p)$ action, let us show it is preserved by the generators of $GL_2(\mathbb{F}_p)$, for example transformations

$$A : x_1 \mapsto x_2, x_2 \mapsto x_1,$$

$$B : x_1 \mapsto ax_1, x_2 \mapsto x_2, a \in \mathbb{F}_p,$$

$$C : x_1 \mapsto x_1 + x_2, x_2 \mapsto x_2.$$

This proof is direct. The action of A corresponds to transposing \mathbf{A}' , which preserves the determinant. The action of B can be calculated as multiplying the entry of \mathbf{A}' in the $(i+1)$ -th row and $(k+1)$ -st column (the ones corresponding to the coefficient of $y_1^{p-1-i} y_2^i$ in v_k) by a^{i-k} . Factoring out a^i from $i+1$ -th row and a^{-k} from the $k+1$ -st column, for all rows and columns, we get the determinant we started from multiplied by $a^{(0+1+\dots+p-1)-(0+1+\dots+p-1)} = 1$. The action of C produces a new matrix, which can be reduced to \mathbf{A}' by a series of row and column operations which do not change the determinant. More precisely, it is possible to use row operations, adding to each row a linear combination of the rows below it, and achieve that the first column is that of the original matrix, and to follow that by a series of column

operations, adding to each column a linear combination of the ones before it, and get the matrix we started from.

We concluded that the determinant of \mathbf{A}' is a polynomial in x_1, x_2 of degree $p(p-1)$, invariant under the $GL_2(\mathbb{F}_p)$ action. Hence, it is a multiple of Q_1 . The coefficient of $x_1^{p(p-1)}$ in both the determinant and Q_1 is equal to 1, and this finishes the proof of lemma. \square

We return to the proof of the proposition and to the system of linear equations

$$\mathbf{A} \vec{h} = Q_l \vec{f}.$$

The inverse of \mathbf{A} is $\frac{1}{\det \mathbf{A}} \tilde{\mathbf{A}}$, for $\tilde{\mathbf{A}}$ the adjugate matrix to \mathbf{A} . For fixed \vec{f} , the unique rational solution \vec{h} is given by

$$\vec{h} = \frac{1}{Q_1} \tilde{\mathbf{A}} Q_l \vec{f}.$$

This solution will be polynomial if and only if every entry of the vector $Q_l \vec{f}$ is divisible (in $S\mathfrak{h}^*$) by Q_1 .

If $l = 1$, this becomes

$$\vec{h} = \tilde{\mathbf{A}} \vec{f},$$

so a polynomial solution exists for every $f \in S\mathfrak{h}^* \otimes S^{p-1}\mathfrak{h}$. In particular, we can pick $f \in S^0\mathfrak{h}^* \otimes S^{p-1}\mathfrak{h}$, and it follows that $Q_1 \otimes S^{p-1}\mathfrak{h}$ is contained in V .

If $l = 0$, then using that Q_0 and Q_1 are algebraically independent, it follows that the polynomial solution \vec{h} will exist if and only if every entry of \vec{f} is divisible by Q_1 . So let $\vec{f} = Q_1 \vec{f}'$, and notice that

$$\vec{h} = Q_0 \tilde{\mathbf{A}} \vec{f}'$$

means that every entry of \vec{h} is divisible by Q_0 . From this it follows that the intersection of the submodule generated by $Q_0 \otimes S^{p-1}\mathfrak{h}$ and V is generated by $Q_0 v_0, \dots, Q_0 v_{p-1}$, which are in degree $(p^2 - 1)(p - 1)$. \square

As explained above, the purpose of proving the previous proposition was to conclude:

Corollary 7.5. *Let M be the quotient of $M_{0,c}(S^i\mathfrak{h})$ by the $H_{0,c}(GL_2(\mathbb{F}_p), \mathfrak{h})$ -submodule generated by singular vectors v_0, \dots, v_{p-1} in degree $p-1$, $Q_1 \otimes S^i\mathfrak{h}$ in degree $p^2 - p$ and $Q_0 \otimes S^i\mathfrak{h}$ in degree $p^2 - 1$. Then its character is*

$$\chi_M(z) = \chi_{M_{0,c}(S^{p-1}\mathfrak{h})}(z)(1 - z^{p-1})(1 - z^{p^2-1})$$

and its Hilbert series is a polynomial

$$\text{Hilb}_M(z) = p \frac{(1 - z^{p-1})(1 - z^{p^2-1})}{(1 - z)^2}.$$

Proposition 7.6. $L_{0,c}(S^{p-1}\mathfrak{h}) = M$.

Proof. By Lemma 4.6, the irreducible representation $L_{0,c}(S^{p-1}\mathfrak{h})$ forms a block of size one. That means that all the irreducible composition factors that appear in the decomposition of $M_{0,c}(S^{p-1}\mathfrak{h})$ and of M , are isomorphic to $L_{0,c}(S^{p-1}\mathfrak{h})[m]$.

As a consequence, the character of $L_{0,c}(S^{p-1}\mathfrak{h})$ divides the character of M ;

$$\chi_{L_{0,c}(S^{p-1}\mathfrak{h})}(z)F(z) = \chi_M(z),$$

for some polynomial $F(z)$ with positive integer coefficients. The character of $L_{0,c}(S^{p-1}\mathfrak{h})$ is of the form

$$\chi_{L_{0,c}(S^{p-1}\mathfrak{h})}(z) = \chi_{M_{0,c}(S^{p-1}\mathfrak{h})}(z)\bar{h}(z)$$

for some polynomial \bar{h} with integer coefficients (\bar{h} is divisible by $(1-z)^2$, as $L_{0,c}(S^{p-1})$ is finite dimensional and $M_{0,c}(S^{p-1})$ has quadratic growth). Substituting this and the character formula for M in the above equation, we get that

$$\bar{h}(z)F(z) = (1-z^{p-1})(1-z^{p^2-1}).$$

Let us define another version of the character which will enable us to compute \bar{h} . For $V = \sum_k V_k$ a graded Cherednik algebra module, we define \tilde{ch}_V to be a function of a formal variable z and of a group element g , defined as

$$\tilde{ch}_V(z, g) = \sum_k z^k \text{tr}|_{V_k}(g).$$

It is then easy to see that

$$\tilde{ch}_{M_{0,c}(S^{p-1}\mathfrak{h})}(z, g) = \text{tr}|_{S^{p-1}\mathfrak{h}}(g) \cdot \frac{1}{\det_{\mathfrak{h}^*}(1-zg)},$$

so

$$\tilde{ch}_M(z, g) = \text{tr}|_{S^{p-1}\mathfrak{h}}(g) \cdot \frac{(1-z^{p-1})(1-z^{p^2-1})}{\det_{\mathfrak{h}^*}(1-zg)}$$

$$\tilde{ch}_{L_{0,c}(S^{p-1}\mathfrak{h})}(z, g) = \text{tr}|_{S^{p-1}\mathfrak{h}}(g) \cdot \frac{\bar{h}(z)}{\det_{\mathfrak{h}^*}(1-zg)}.$$

Let $g \in GL_2(\mathbb{F}_p)$. It can be put to Jordan form over a quadratic extension \mathbb{F}_q of \mathbb{F}_p , and assume it is diagonalizable with different eigenvalues, of the form

$$\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

with $\lambda \neq \mu \in \mathbb{F}_q$. Then

$$\text{tr}|_{S^{p-1}\mathfrak{h}}(g) = \lambda^{p-1} + \lambda^{p-2}\mu + \dots + \mu^{p-1} = \frac{\lambda^p - \mu^p}{\lambda - \mu} \neq 0$$

and $\tilde{ch}_{L_{0,c}(S^{p-1}\mathfrak{h})}(z, g)$ is a polynomial in z , so

$$\frac{\bar{h}(z)}{\det_{\mathfrak{h}^*}(1-zg)} = \frac{\bar{h}(z)}{(1-z\lambda^{-1})(1-z\mu^{-1})}$$

must be a polynomial in z as well. By choosing all possible λ and μ in $\mathbb{F}_p \subseteq \mathbb{F}_q$, this implies that $\bar{h}(z)$ is divisible by all linear polynomials of the form $1-z\lambda^{-1}$, and hence by their product $1-z^{p-1}$. If λ and μ are in the extension \mathbb{F}_q and not in \mathbb{F}_p , then the product $(1-z\lambda^{-1})(1-z\mu^{-1})$ is an irreducible quadratic polynomial with coefficients in \mathbb{F}_p with a constant term 1. All such polynomials can be obtained in this way, and $\bar{h}(z)$ is divisible by their product $(1-z^{p^2-1})/(1-z^{p-1})$. From this we conclude that $\bar{h}(z)$ is divisible by $1-z^{p^2-1}$.

Let us write

$$\bar{h}(z) = (1-z^{p^2-1})\phi(z)$$

for some polynomial ϕ . Then

$$\phi(z)F(z) = 1 - z^{p-1}.$$

However, it follows from Lemma 7.2 that \bar{h} is of the form $1 - z^{p-1} + \dots$, so $\phi(z)$ is of that form as well, and it follows that $\phi(z) = 1 - z^{p-1}$, $F(z) = 1$ and $L_{0,c}(S^{p-1}\mathfrak{h}) = M$. \square

7.2. Characters of $L_{t,c}(S^i\mathfrak{h})$ for $i = p - 1$ and $t = 1$. Computing the character of $L_{1,c}(S^{p-1}\mathfrak{h})$ is very similar to computing the character of $L_{0,c}(S^{p-1}\mathfrak{h})$ in the previous section. We define a set of vectors analogous to v_i :

$$v'_k = \sum_{i=0}^k (-1)^i x_1^{p(k-i)} x_2^{p(p-1-k+i)} \otimes y_1^{p-1-i} y_2^i + \sum_{i=k}^{p-1} (-1)^i x_1^{p(p-1+k-i)} x_2^{p(i-k)} \otimes y_1^{p-1-i} y_2^i.$$

Lemma 7.7. *The space $\text{span}_{\mathbb{K}}\{v'_0, \dots, v'_{p-1}\} \subseteq S^{p(p-1)}\mathfrak{h}^* \otimes S^{p-1}\mathfrak{h} \cong M_{1,c}(S^{p-1}\mathfrak{h})_{p(p-1)}$ consists of singular vectors, and isomorphic to $S^{p-1}\mathfrak{h}$ as a $GL_2(\mathbb{F}_p)$ representation. This is the only space of singular vectors in $M_{1,c}(S^{p-1}\mathfrak{h})_{p \cdot i}$ for $i = 1, \dots, p-1$.*

Proof. The proof that they are singular is an explicit computation analogous to the one in the proof of Lemma 7.1, showing that one vector from this irreducible representation of $GL_2(\mathbb{F}_p)$ is singular. The space spanned by them is only space of p -th powers in degrees $p, 2p, \dots, (p-1)p$ which is isomorphic to $S^{p-1}\mathfrak{h}$ as a $GL_2(\mathbb{F}_p)$ representation; this follows directly from Lemma 7.2 and implies that this is the only space of singular vectors for generic c in degrees up to $p(p-1)$. \square

Proposition 7.8. *Let M' be the quotient of $M_{1,c}(S^i\mathfrak{h})$ by the $H_{1,c}(GL_2(\mathbb{F}_p), \mathfrak{h})$ -submodule generated by singular vectors v'_0, \dots, v'_{p-1} in degree $p(p-1)$, $Q_1^p \otimes S^i\mathfrak{h}$ in degree $p(p^2 - p)$ and $Q_0^p \otimes S^i\mathfrak{h}$ in degree $p(p^2 - 1)$. Its character is*

$$\chi_{M'}(z) = \chi_{M_{1,c}(S^{p-1}\mathfrak{h})}(z^p)(1 - z^{p(p-1)})(1 - z^{p(p^2-1)}) \left(\frac{1 - z^p}{1 - z} \right)^2$$

and the Hilbert series

$$\text{Hilb}_{M'}(z) = p \frac{(1 - z^{p(p-1)})(1 - z^{p(p^2-1)})}{(1 - z)^2}.$$

Proof. The claim is equivalent to the reduced character being equal to

$$\chi_{M_{1,c}(S^{p-1}\mathfrak{h})}(z)(1 - z^{p-1})(1 - z^{p^2-1}).$$

By definition of M' and the reduced character, it is equal to the character of the $S\mathfrak{h}^*$ -module defined as the quotient of $S\mathfrak{h}^* \otimes S^{p-1}\mathfrak{h}$ by v_0, \dots, v_{p-1} from the previous section, $Q_0 \otimes S^i\mathfrak{h}$ and $Q_1 \otimes S^i\mathfrak{h}$. Corollary 7.5 in the previous section shows that the character of this module is as claimed in the proposition. \square

Finally, we have

Proposition 7.9. *For generic c , $L_{1,c}(S^{p-1}\mathfrak{h}) = M'$.*

Proof. The character of $L_{1,c}(S^{p-1}\mathfrak{h})$ for generic c is of the form

$$\chi_{L_{1,c}(S^{p-1}\mathfrak{h})}(z) = \chi_{M_{1,c}(S^{p-1}\mathfrak{h})}(z^p) \left(\frac{1 - z^p}{1 - z} \right)^2 \bar{h}'(z^p)$$

for some polynomial $\overline{h'}$. It divides the character of M' , so $\overline{h'}(z)$ divides $(1 - z^{p-1})(1 - z^{p^2-1})$. Using the same version of the character as in the proof of Proposition 7.6, we see that

$$\tilde{ch}_{L_{1,c}(S^{p-1}\mathfrak{h})}(z, g) = \text{tr}|_{S^{p-1}\mathfrak{h}}(g) \cdot \frac{\overline{h'}(z^p)}{\det_{\mathfrak{h}^*}(1 - zg)},$$

and we see that $h(z^p)$ is divisible by $(1 - z^{p^2-1})$. From this it follows that $h(z^p)$ is divisible by $(1 - z^{p(p^2-1)})$. Finally, it follows from the previous proposition that $\overline{h'}(z)$ is of the form $1 - z^{p-1} + \dots$, and from this, its divisibility by $(1 - z^{p^2-1})$ and the fact it divides $(1 - z^{p^2-1})(1 - z^{p-1})$ it follows $\overline{h'}(z) = (1 - z^{p^2-1})(1 - z^{p-1})$ and $L_{1,c}(S^{p-1}\mathfrak{h}) = M'$. □

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